Arithmetic of algebraic fractions

1.4

Introduction

Just as one whole number divided by another is called a numerical fraction, so one algebraic expression divided by another is known as an **algebraic fraction**. Examples of the latter are

$$\frac{x}{y}, \quad \frac{3x+2y}{x-y}, \qquad \text{and} \qquad \frac{x^2+3x+1}{x-4}$$

In this section we explain how algebraic fractions can be simplified, added, subtracted, multiplied and divided.

Prerequisites

Before starting this Section you should . . .

① be familiar with the arithmetic of numerical fractions

Learning Outcomes

After completing this Section you should be able to . . .

✓ add, subtract, multiply and divide algebraic fractions

1. Cancelling common factors

Consider the fraction $\frac{10}{35}$. To simplify it we can factorise the numerator and the denominator and then cancel any common factors. Common factors are those factors which occur in both the numerator and the denominator. Thus

$$\frac{10}{35} = \frac{\cancel{5} \times 2}{7 \times \cancel{5}} = \frac{2}{7}$$

Note that the common factor 5 has been cancelled. It is important to remember that only *common factors* can be cancelled. The fractions $\frac{10}{35}$ and $\frac{2}{7}$ have identical values - they are equivalent fractions - but $\frac{2}{7}$ is in a simpler form than $\frac{10}{35}$.
We apply the same process when simplifying algebraic fractions.

Example Simplify, if possible,

a) $\dfrac{yx}{2x}$, b) $\dfrac{x}{xy}$, c) $\dfrac{x}{x+y}$

Solution

(a) In the expression $\dfrac{yx}{2x}$, x is a factor common to both numerator and denominator. This common factor can be cancelled to give

$$\frac{y\,\cancel{x}}{2\,\cancel{x}} = \frac{y}{2}$$

(b) Note that $\frac{x}{xy}$ can be written $\frac{1x}{xy}$. The common factor of x can be cancelled to give

$$\frac{1.\,\cancel{x}}{\cancel{x}y} = \frac{1}{y}$$

(c) In the expression $\frac{x}{x+y}$ notice that an x appears in both numerator and denominator. However x is not a common factor. Recall that factors of an expression are *multiplied* together whereas in the denominator x is *added* to y. This expression cannot be simplified.

Simplify, if possible, a) $\frac{abc}{3ac}$, b) $\frac{3ab}{b+a}$
When simplifying remember only common factors can be cancelled.

Your solution

a) $\frac{abc}{3ac} =$ b) $\frac{3ab}{b+a} =$

a) $\frac{b}{3}$ b) This cannot be simplified

Example Simplify a) $\frac{21x^3}{14x}$, b) $\frac{36x}{12x^3}$

Solution

Factorising and cancelling common factors gives:

$$\text{a)}\quad \frac{21x^3}{14x} = \frac{\cancel{7}\times 3\times \cancel{x}\times x^2}{\cancel{7}\times 2\times \cancel{x}} = \frac{3x^2}{2} \qquad\qquad \text{b)}\quad \frac{36x}{12x^3} = \frac{12\times 3\times x}{12\times x\times x^2} = \frac{3}{x^2}$$

Example Simplify $\frac{3x+6}{6x+12}$.

Solution

First we factorise the numerator and the denominator to see if there are any common factors.

$$\frac{3x+6}{6x+12} = \frac{3(x+2)}{6(x+2)} = \frac{3}{6} = \frac{1}{2}$$

The factors $x+2$ and 3 have been cancelled.

Simplify $\frac{12}{2x+8}$.

Your solution

$\frac{12}{2x+8} =$

Factorise the numerator and denominator, and cancel any common factors. $\frac{6\times 2}{2(x+4)} = \frac{6}{x+4}$

Example Show that the algebraic fraction $\frac{3}{x+1}$ and $\frac{3(x+4)}{x^2+5x+4}$ are equivalent.

Solution

The denominator, x^2+5x+4, can be factorised as $(x+1)(x+4)$ so that

$$\frac{3(x+4)}{x^2+5x+4} = \frac{3(x+4)}{(x+1)(x+4)}$$

Note that $(x+4)$ is a factor common to both the numerator and the denominator and can be cancelled to leave $\frac{3}{x+1}$. Thus $\frac{3}{x+1}$ and $\frac{3(x+4)}{x^2+5x+4}$ are equivalent fractions.

Show that $\frac{1}{x-1}$ and $\frac{x-1}{x^2-2x+1}$ are equivalent fractions.
First factorise the denominator

Your solution
$x^2 - 2x + 1$:

$(x-1)(x-1)$

Identify the factor which is common to both numerator and denominator and cancel this common factor.

Your solution
$\frac{x-1}{(x-1)(x-1)} =$

$\frac{1}{x-1}$

Hence the two given fractions are equivalent.

Example Simplify

$$\frac{6(4-8x)(x-2)}{1-2x}$$

Solution

The factor $4-8x$ can be factorised to $4(1-2x)$. Thus

$$\frac{6(4-8x)(x-2)}{1-2x} = \frac{(6)(4)(1-2x)(x-2)}{(1-2x)} = 24(x-2)$$

Simplify $\frac{x^2+2x-15}{2x^2-5x-3}$
First factorise the numerator and the denominator.

Your solution
$\frac{x^2+2x-15}{2x^2-5x-3} =$

$\frac{(x+5)(x-3)}{(2x+1)(x-3)}$

Finally cancel any common factors to leave

Your solution

$\frac{x+5}{2x+1}$

Exercises

1. Simplify, if possible,

   a) $\frac{19}{38}$, b) $\frac{14}{28}$, c) $\frac{35}{40}$, d) $\frac{7}{11}$, e) $\frac{14}{56}$

2. Simplify, if possible, a) $\frac{14}{21}$, b) $\frac{36}{96}$, c) $\frac{13}{52}$, d) $\frac{52}{13}$

3. Simplify a) $\frac{5z}{z}$, b) $\frac{25z}{5z}$, c) $\frac{5}{25z^2}$, d) $\frac{5z}{25z^2}$

4. Simplify

   a) $\frac{4x}{3x}$, b) $\frac{15x}{x^2}$, c) $\frac{4s}{s^3}$, d) $\frac{21x^4}{7x^3}$

5. Simplify, if possible,

   a) $\frac{x+1}{2(x+1)}$, b) $\frac{x+1}{2x+2}$, c) $\frac{2(x+1)}{x+1}$, d) $\frac{3x+3}{x+1}$, e) $\frac{5x-15}{5}$, f) $\frac{5x-15}{x-3}$.

6. Simplify, if possible,

   a) $\frac{5x+15}{25x+5}$, b) $\frac{5x+15}{25x}$, c) $\frac{5x+15}{25}$, d) $\frac{5x+15}{25x+1}$

7. Simplify

   a) $\frac{x^2+10x+9}{x^2+8x-9}$, b) $\frac{x^2-9}{x^2+4x-21}$, c) $\frac{2x^2-x-1}{2x^2+5x+2}$, d) $\frac{3x^2-4x+1}{x^2-x}$, e) $\frac{5z^2-20z}{2z-8}$

8. Simplify a) $\frac{6}{3x+9}$, b) $\frac{2x}{4x^2+2x}$, c) $\frac{3x^2}{15x^3+10x^2}$

9. Simplify a) $\frac{x^2-1}{x^2+5x+4}$, b) $\frac{x^2+5x+6}{x^2+x-6}$.

Answers

1. a) $\frac{1}{2}$, b) $\frac{1}{2}$, c) $\frac{7}{8}$, d) $\frac{7}{11}$, e) $\frac{1}{4}$.
2. a) $\frac{2}{3}$, b) $\frac{3}{8}$, c) $\frac{1}{4}$, d) 4
3. a) 5 b) 5, c) $\frac{1}{5z^2}$, d) $\frac{1}{5z}$.
4. a) $\frac{4}{3}$, b) $\frac{15}{x}$, c) $\frac{4}{s^2}$, d) $3x$
5. a) $\frac{1}{2}$, b) $\frac{1}{2}$, c) 2, d) 3, e) $x-3$, f) 5
6. a) $\frac{x+3}{5x+1}$, b) $\frac{x+3}{5x}$, c) $\frac{x+3}{5}$, d) $\frac{5(x+3)}{25x+1}$
7. a) $\frac{x+1}{x-1}$, b) $\frac{x+3}{x+7}$, c) $\frac{x-1}{x+2}$, d) $\frac{3x-1}{x}$, e) $\frac{5z}{2}$
8. a) $\frac{2}{x+3}$, b) $\frac{1}{2x+1}$, c) $\frac{3}{5(3x+2)}$.
9. a) $\frac{x-1}{x+4}$, b) $\frac{x+2}{x-2}$.

2. Multiplication and division of algebraic fractions

To multiply two fractions (numerical or algebraic) we multiply their numerators together and then multiply their denominators together. That is

Key Point

Multiplication:

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

Any factors common to both numerator and denominator can be cancelled. This cancellation can be performed before or after the multiplication.

Division is performed by inverting the second fraction and then multiplying.

Key Point

Division:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

Example Simplify a) $\frac{2a}{c} \times \frac{4}{c}$, b) $\frac{2a}{c} \times \frac{c}{4}$, c) $\frac{2a}{c} \div \frac{4}{c}$

Solution

(a)

$$\frac{2a}{c} \times \frac{4}{c} = \frac{8a}{c^2}$$

(b)

$$\frac{2a}{c} \times \frac{c}{4} = \frac{2ac}{4c} = \frac{2a}{4} = \frac{a}{2}$$

(c) Division is performed by inverting the second fraction and then multiplying.

$$\frac{2a}{c} \div \frac{4}{c} = \frac{2a}{c} \times \frac{c}{4} = \frac{a}{2} \qquad \text{from the result in b)}$$

Example Simplify a) $\frac{1}{5x} \times 3x$, b) $\frac{1}{x} \times x$, c) $\frac{1}{y} \times x$, d) $\frac{y}{x} \times x$.

Solution

a) Note that $3x = \frac{3x}{1}$. Then $\dfrac{1}{5x} \times 3x = \dfrac{1}{5x} \times \dfrac{3x}{1} = \dfrac{3x}{5x} = \dfrac{3}{5}$

b) x can be written as $\frac{x}{1}$. Then $\dfrac{1}{x} \times x = \dfrac{1}{x} \times \dfrac{x}{1} = \dfrac{x}{x} = 1$

c) $\dfrac{1}{y} \times x = \dfrac{1}{y} \times \dfrac{x}{1} = \dfrac{x}{y}$

d) $\dfrac{y}{x} \times x = \dfrac{y}{x} \times \dfrac{x}{1} = \dfrac{yx}{x} = y$

Example Simplify $\dfrac{\frac{2x}{y}}{\frac{3x}{2y}}$

Solution

We can write the fraction as $\frac{2x}{y} \div \frac{3x}{2y}$. Inverting the second fraction and multiplying we find

$$\frac{2x}{y} \times \frac{2y}{3x} = \frac{4xy}{3xy} = \frac{4}{3}$$

Example Simplify $\dfrac{4x+2}{x^2+4x+3} \times \dfrac{x+3}{7x+5}$

Solution

Factorising the numerator and denominator we find

$$\frac{4x+2}{x^2+4x+3} \times \frac{x+3}{7x+5} = \frac{2(2x+1)}{(x+1)(x+3)} \times \frac{x+3}{7x+5} = \frac{2(2x+1)(x+3)}{(x+1)(x+3)(7x+5)} = \frac{2(2x+1)}{(x+1)(7x+5)}$$

It is usually better to factorise first and cancel any common factors before multiplying. *Don't remove any brackets unnecessarily otherwise common factors will be difficult to spot.*

Example Simplify

$$\frac{15}{3x-1} \div \frac{3}{2x+1}$$

Solution

To divide we invert the second fraction and multiply:

$$\frac{15}{3x-1} \div \frac{3}{2x+1} = \frac{15}{3x-1} \times \frac{2x+1}{3} = \frac{(5)(3)(2x+1)}{3(3x-1)} = \frac{5(2x+1)}{3x-1}$$

Exercises

1. Simplify a) $\frac{5}{9} \times \frac{3}{2}$, b) $\frac{14}{3} \times \frac{3}{9}$, c) $\frac{6}{11} \times \frac{3}{4}$, d) $\frac{4}{7} \times \frac{28}{3}$
2. Simplify a) a) $\frac{5}{9} \div \frac{3}{2}$, b) $\frac{14}{3} \div \frac{3}{9}$, c) $\frac{6}{11} \div \frac{3}{4}$, d) $\frac{4}{7} \div \frac{28}{3}$
3. Simplify

   a) $2 \times \frac{x+y}{3}$, b) $\frac{1}{3} \times 2(x+y)$, c) $\frac{2}{3} \times (x+y)$
4. Simplify

   a) $3 \times \frac{x+4}{7}$, b) $\frac{1}{7} \times 3(x+4)$, c) $\frac{3}{7} \times (x+4)$, d) $\frac{x}{y} \times \frac{x+1}{y+1}$, e) $\frac{1}{y} \times \frac{x^2+x}{y+1}$ f) $\frac{\pi d^2}{4} \times \frac{Q}{\pi d^2}$,

   g) $\frac{Q}{\pi d^2/4}$
5. Find $\frac{6/7}{s+3}$.
6. Find $\frac{3}{x+2} \div \frac{x}{2x+4}$.
7. Find $\frac{5}{2x+1} \div \frac{x}{3x-1}$.

Answers

1. a) $\frac{5}{6}$, b) $\frac{14}{9}$, c) $\frac{9}{22}$, d) $\frac{16}{3}$.
2. a) $\frac{10}{27}$, b) 14, c) $\frac{8}{11}$, d) $\frac{3}{49}$.
3. a) $\frac{2(x+y)}{3}$, b) $\frac{2(x+y)}{3}$, c) $\frac{2(x+y)}{3}$
4. a) $\frac{3(x+4)}{7}$, b) $\frac{3(x+4)}{7}$, c) $\frac{3(x+4)}{7}$, d) $\frac{x(x+1)}{y(y+1)}$, e) $\frac{x(x+1)}{y(y+1)}$, f) $Q/4$, g) $\frac{4Q}{\pi d^2}$
5. $\frac{6}{7(s+3)}$.
6. $\frac{6}{x}$. 7. $\frac{5(3x-1)}{x(2x+1)}$

3. Addition and subtraction of algebraic fractions

To add two algebraic fractions the **lowest common denominator** must be found first. This is the simplest algebraic expression that has the given denominators as its factors. All fractions must be written with this lowest common denominator. Their sum is found by adding the numerators and dividing the result by the lowest common denominator.

To subtract two fractions the process is similar. The fractions are written with the lowest common denominator. The difference is found by subtracting the numerators and dividing the result by the lowest common denominator.

Example State the simplest expression which has $x+1$ and $x+4$ as its factors.

Solution

The simplest expression is $(x+1)(x+4)$. Note that both $x+1$ and $x+4$ are factors.

Example State the simplest expression which has $x-1$ and $(x-1)^2$ as its factors.

Solution

The simplest expression is $(x-1)^2$. Clearly $(x-1)^2$ must be a factor of this expression. Also, because we can write $(x-1)^2 = (x-1)(x-1)$ it follows that $x-1$ is a factor too.

Example Express as a single fraction $\dfrac{3}{x+1} + \dfrac{2}{x+4}$

Solution

The simplest expression which has both denominators as its factors is $(x+1)(x+4)$. This is the lowest common denominator. Both fractions must be written using this denominator. Note that $\frac{3}{x+1}$ is equivalent to $\frac{3(x+4)}{(x+1)(x+4)}$ and $\frac{2}{x+4}$ is equivalent to $\frac{2(x+1)}{(x+1)(x+4)}$. Thus writing both fractions with the same denominator we have

$$\frac{3}{x+1} + \frac{2}{x+4} = \frac{3(x+4)}{(x+1)(x+4)} + \frac{2(x+1)}{(x+1)(x+4)}$$

The sum is found by adding the numerators and dividing the result by the lowest common denominator.

$$\frac{3(x+4)}{(x+1)(x+4)} + \frac{2(x+1)}{(x+1)(x+4)} = \frac{3(x+4)+2(x+1)}{(x+1)(x+4)} = \frac{5x+14}{(x+1)(x+4)}$$

Key Point

Addition: To add two fractions

(i) find the lowest common denominator

(ii) express each fraction with this denominator

(iii) Add the numerators and divide the result by the lowest common denominator

Example Express $\frac{1}{x-1} + \frac{5}{(x-1)^2}$ as a single fraction.

Solution

The simplest expression having both denominators as its factors is $(x-1)^2$. We write both fractions with this denominator.

$$\frac{1}{x-1} + \frac{5}{(x-1)^2} = \frac{x-1}{(x-1)^2} + \frac{5}{(x-1)^2} = \frac{x-1+5}{(x-1)^2} = \frac{x+4}{(x-1)^2}$$

Find $\frac{3}{x+7} + \frac{5}{x+2}$.
First find the lowest common denominator:

Your solution

$(x+7)(x+2)$

Both fractions are re-written using this lowest common denominator:

Your solution
$\frac{3}{x+7} + \frac{5}{x+2} =$

$\frac{3(x+2)}{(x+7)(x+2)} + \frac{5(x+7)}{(x+7)(x+2)}$

Add the numerators and simplify:

Your solution
$\frac{3}{x+7} + \frac{5}{x+2} =$

$\frac{8x+41}{(x+7)(x+2)}$

Example Find $\frac{5x}{7} - \frac{3x-4}{2}$.

Solution

In this example both denominators are simply numbers. The lowest common denominator is 14, and both fractions are re-written with this denominator. Thus

$$\frac{10x}{14} - \frac{7(3x-4)}{14} = \frac{10x - 7(3x-4)}{14} = \frac{28 - 11x}{14}$$

Example Find $\frac{1}{x} + \frac{1}{y}$.

Solution

The simplest expression which has x and y as its factors is xy. This is the lowest common denominator. Both fractions are written using this denominator. Noting that $\frac{1}{x} = \frac{y}{xy}$ and that $\frac{1}{y} = \frac{x}{xy}$ we find

$$\frac{1}{x} + \frac{1}{y} = \frac{y}{xy} + \frac{x}{xy} = \frac{y+x}{xy}$$

No cancellation is now possible because neither x nor y is a factor of the numerator.

Exercises

1. Find a) $\frac{x}{4}+\frac{x}{7}$, b) $\frac{2x}{5}+\frac{x}{9}$, c) $\frac{2x}{3}-\frac{3x}{4}$, d) $\frac{x}{x+1}-\frac{2}{x+2}$, e) $\frac{x+1}{x}+\frac{3}{x+2}$, f) $\frac{2x+1}{3}-\frac{x}{2}$, g) $\frac{x+3}{2x+1}-\frac{x}{3}$, h) $\frac{x}{4}-\frac{x}{5}$
2. Find

   a) $\frac{1}{x+2}+\frac{2}{x+3}$, b) $\frac{2}{x+3}+\frac{5}{x+1}$, c) $\frac{2}{2x+1}-\frac{3}{3x+2}$, d) $\frac{x+1}{x+3}+\frac{x+4}{x+2}$, e) $\frac{x-1}{x-3}+\frac{x-1}{(x-3)^2}$.
3. Find $\frac{5}{2x+3}+\frac{4}{(2x+3)^2}$.
4. Find $\frac{1}{7}s+\frac{11}{21}$
5. Express $\frac{A}{2x+3}+\frac{B}{x+1}$ as a single fraction.
6. Express $\frac{A}{2x+5}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2}$ as a single fraction.
7. Express $\frac{A}{x+1}+\frac{B}{(x+1)^2}$ as a single fraction.
8. Express $\frac{Ax+B}{x^2+x+10}+\frac{C}{x-1}$ as a single fraction.
9. Express $Ax+B+\frac{C}{x+1}$ as a single fraction.
10. Show that $\frac{x_1}{\frac{1}{x_3}-\frac{1}{x_2}}$ is equal to $\frac{x_1x_2x_3}{x_2-x_3}$.
11. Find a) $\frac{3x}{4}-\frac{x}{5}+\frac{x}{3}$, b) $\frac{3x}{4}-\left(\frac{x}{5}+\frac{x}{3}\right)$.

Answers

1. a) $\frac{11x}{28}$, b) $\frac{23x}{45}$, c) $-\frac{x}{12}$, d) $\frac{x^2-2}{(x+1)(x+2)}$, e) $\frac{x^2+6x+2}{x(x+2)}$, f) $\frac{x+2}{6}$, g) $\frac{9+2x-2x^2}{3(2x+1)}$, h) $\frac{x}{20}$.
2. a) $\frac{3x+7}{(x+2)(x+3)}$, b) $\frac{7x+17}{(x+3)(x+1)}$, c) $\frac{1}{(2x+1)(3x+2)}$, d) $\frac{2x^2+10x+14}{(x+3)(x+2)}$, e) $\frac{x^2-3x+2}{(x-3)^2}$
3. $\frac{10x+19}{(2x+3)^2}$
4. $\frac{3s+11}{21}$.
5. $\frac{A(x+1)+B(2x+3)}{(2x+3)(x+1)}$.
6. $\frac{A(x-1)^2+B(x-1)(2x+5)+C(2x+5)}{(2x+5)(x-1)^2}$
7. $\frac{A(x+1)+B}{(x+1)^2}$
8. $\frac{(Ax+B)(x-1)+C(x^2+x+10)}{(x-1)(x^2+x+10)}$
9. $\frac{(Ax+B)(x+1)+C}{x+1}$
11. a) $\frac{53x}{60}$, b) $\frac{13x}{60}$.