De Moivre's Theorem 10.4

Introduction

In this Section we introduce De Moivre's theorem and examine some of its consequences. We shall see that one of its uses is in obtaining relationships between trigonometric functions of multiple angles (like $\sin 3x$, $\cos 7x$ etc) and powers of trigonometric functions (like $\sin^2 x$, $\cos^4 x$ etc). Another important aspect of De Moivre's theorem lies in its use in obtaining complex roots of polynomial equations. In this application we re-examine our definition of the argument $\arg(z)$ of a complex number.

Prerequisites

Before starting this Section you should ...

① be familiar with the polar form

② be familiar with the Argand diagram

③ be familiar with the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$

④ know how to expand $(x+y)^n$ when n is a positive integer

Learning Outcomes

After completing this Section you should be able to ...

✓ employ De Moivre's theorem in a number of applications

✓ understand more clearly the argument $\arg(z)$ of a complex number

✓ obtain complex roots of complex numbers

1. De Moivre's Theorem

We have seen, in Section 10.2, that, in polar form, if $z = r(\cos\theta + \mathrm{i}\sin\theta)$ and $w = t(\cos\phi + \mathrm{i}\sin\phi)$ then the product zw is easily obtained:

$$zw = rt(\cos(\theta+\phi) + \mathrm{i}\sin(\theta+\phi))$$

In particular, if $r = 1, t = 1$ and $\theta = \phi$ (i.e. $z = w = \cos\theta + \mathrm{i}\sin\theta$), we obtain

$$(\cos\theta + \mathrm{i}\sin\theta)^2 = \cos 2\theta + \mathrm{i}\sin 2\theta$$

Multiplying each side by $\cos\theta + \mathrm{i}\sin\theta$ gives

$$(\cos\theta + \mathrm{i}\sin\theta)^3 = (\cos 2\theta + \mathrm{i}\sin 2\theta)(\cos\theta + \mathrm{i}\sin\theta) = (\cos 3\theta + \mathrm{i}\sin 3\theta)$$

on adding the arguments of the terms in the product.
Similarly

$$(\cos\theta + \mathrm{i}\sin\theta)^4 = (\cos 4\theta + \mathrm{i}\sin 4\theta).$$

After completing p such products we have:

$$(\cos\theta + \mathrm{i}\sin\theta)^p = \cos p\theta + \mathrm{i}\sin p\theta$$

where p is a positive integer.
In fact this result can be shown to be true for those cases in which p is a negative integer and even when p is a rational number e.g. $p = \frac{1}{2}$.

Key Point

If p is a rational number:

$$(\cos\theta + \mathrm{i}\sin\theta)^p = \cos p\theta + \mathrm{i}\sin p\theta$$

This result is known as **De Moivre's Theorem**.

In exponential form De Moivre's theorem, in the case when p is a positive integer, is simply a statement of the laws of indices:

$$(\mathrm{e}^{\mathrm{i}\theta})^p = \mathrm{e}^{\mathrm{i}p\theta}$$

Example Use De Moivre's theorem to obtain an expression for $\cos 3\theta$ in terms of powers of $\cos\theta$ alone.

Solution

From De Moivre's theorem (Key Point above with $p = 3$) we have

$$(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos 3\theta + \mathrm{i}\sin 3\theta$$

However, expanding the left-hand side (using: $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$) we have:

$$\cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta = \cos 3\theta + \mathrm{i}\sin 3\theta$$

and then, equating the real parts of both sides, gives the relation:

$$\cos^3\theta - 3\cos\theta\sin^2\theta = \cos 3\theta$$

or, replacing $\sin^2\theta$ by $(1-\cos^2\theta)$:

$$\cos^3\theta - 3\cos\theta(1-\cos^2\theta) = \cos 3\theta$$

Finally:

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

is the required relation.

Use the last example to find an expression for $\sin 3\theta$ in terms of powers of $\sin\theta$ alone.

Your solution

You should obtain $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ since, from the previous example (but this time equating imaginary parts of both sides)

$$\begin{aligned}\sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta\\ &= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta\\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

Without using tables or a calculator obtain the value of $\sin 60^0$ given that $\sin 20^0 \approx 0.342020$

Your solution

You should obtain $\sin 60^0 \approx 0.866025$ since, from the previous example, choosing $\theta = 20^0$ we obtain:

$$\sin 60^0 = 3\sin 20^0 - 4\sin^3 20^0 \approx 0.866025 \qquad (= \frac{\sqrt{3}}{2} \quad \text{exactly})$$

2. De Moivre's Theorem and Root Finding

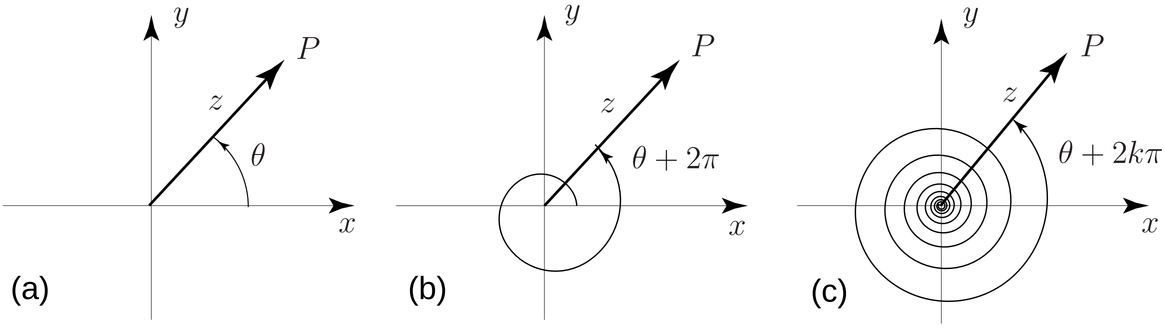
In this Section we ask if we can obtain fractional powers of complex numbers; for example what are the values of $8^{1/3}$ or $(-24)^{1/4}$ or even $(1 + \mathrm{i})^{1/2}$?
More precisely, for these three examples, we are asking for those values of z which satisfy

$$z^3 - 8 = 0 \qquad \text{or} \qquad z^4 + 24 = 0 \qquad \text{or} \qquad z^2 - (1 + \mathrm{i}) = 0$$

Each of these problems involve finding roots of a complex number.
To solve problems such as these we shall need to be more careful with our interpretation of $\arg(z)$ for a given complex number z.

Arg(z) revisited

By definition $\arg(z)$ is the angle made by the line representing z with the positive x-axis. See (a) in the following diagram. However, as the second diagram (b) shows you can increase θ by 2π (or 360^0) and still obtain the *same* line in the xy plane. In general, as indicated in diagram (c) any integer multiple of 2π can be added to or subtracted from $\arg(z)$ without affecting the **Cartesian form** of the complex number.



Key Point

$\arg(z)$ is unique only up to an integer multiple of 2π

For example:

$$z = 1 + \mathrm{i} = \sqrt{2}(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}) \qquad \text{in polar form}$$

However, we could also write, equivalently:

$$z = 1 + \mathrm{i} = \sqrt{2}(\cos(\frac{\pi}{4} + 2\pi) + \mathrm{i}\sin(\frac{\pi}{4} + 2\pi))$$

or, in full generality:

$$z = 1 + \mathrm{i} = \sqrt{2}(\cos(\frac{\pi}{4} + 2k\pi) + \mathrm{i}\sin(\frac{\pi}{4} + 2k\pi)) \qquad k = 0, \pm 1, \pm 2, \cdots$$

This last expression shows that in the polar form of a complex number the argument of z, $\arg(z)$, can assume many different values, each one differing by an integer multiple of 2π. This is nothing more than a consequence of the well-known properties of the trigonometric functions:

$$\cos(\theta + 2k\pi) \equiv \cos\theta, \qquad \sin(\theta + 2k\pi) \equiv \sin\theta \qquad \text{for } any \text{ integer } k$$

We shall now show how we can use this more general interpretation of $\arg(z)$ in the process of finding roots.

Example Find all the values of $8^{1/3}$.

Solution

Solving $z = 8^{1/3}$ for z is equivalent to solving the cubic equation $z^3 - 8 = 0$. We expect that there are three possible values of z satisfying this cubic equation. Thus, rearranging: $z^3 = 8$. Now write the right-hand side as a complex number in polar form:

$$z^3 = 8(\cos 0 + \mathrm{i}\sin 0)$$

(i.e. $r = |8| = 8$ and $\arg(8) = 0$). However, if we now generalise our expression for the argument, by adding an arbitrary integer multiple of 2π, we obtain the modified expression:

$$z^3 = 8(\cos(2k\pi) + \mathrm{i}\sin(2k\pi)) \qquad k = 0, \pm 1, \pm 2, \cdots$$

Now take the cube root of both sides

$$\begin{aligned} z &= \sqrt[3]{8}(\cos(2k\pi) + \mathrm{i}\sin(2k\pi))^{\frac{1}{3}} \\ &= \sqrt[3]{8}(\cos\frac{2k\pi}{3} + \mathrm{i}\sin\frac{2k\pi}{3}) \qquad \text{using De Moivre's theorem.} \end{aligned}$$

Solution (contd.)

Now in this expression k can take any integer value or zero. The normal procedure is to take three consecutive values of k (say $k = 0, 1, 2$). Any other value of k chosen will lead to a root (a value of z) which repeats one of the three already determined.

$$\begin{aligned} \text{So if} \quad k &= 0 \qquad z_0 = 2(\cos 0 + \mathrm{i}\sin 0) = 2 \\ k &= 1 \qquad z_1 = 2(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}) = -1 + \mathrm{i}\sqrt{3} \\ k &= 2 \qquad z_2 = 2(\cos\frac{4\pi}{3} + \mathrm{i}\sin\frac{4\pi}{3}) = -1 - \mathrm{i}\sqrt{3} \end{aligned}$$

These are the three (complex) values of $8^{\frac{1}{3}}$. The reader should verify, by direct multiplication, that $(-1+\mathrm{i}\sqrt{3})^3 = 8$ and that $(-1-\mathrm{i}\sqrt{3})^3 = 8$. The reader may have noticed within this example a subtle change in notation. When we write, for example, $8^{1/3}$ then we are expecting three possible values, as calculated above. However, when we write $\sqrt[3]{8}$ then we are only expecting one value: that delivered by your calculator. Note the complex roots are complex conjugates (since $z^3 - 8 = 0$ is a polynomial equation with real coefficients).

In the above example we have worked with the polar form. Precisely the same calculation can be carried through using the exponential form of a complex number. We take this opportunity to repeat this calculation but working exclusively in exponential form.
Thus

$$\begin{aligned} z^3 &= 8 \\ &= 8\mathrm{e}^{\mathrm{i}(0)} \qquad (\text{i.e. } r = |8| = 8 \quad \text{and} \quad \arg(8) = 0) \\ &= 8\mathrm{e}^{\mathrm{i}(2k\pi)} \qquad k = 0, \pm 1, \pm 2, \cdots \end{aligned}$$

therefore taking cube roots

$$\begin{aligned} z &= \sqrt[3]{8}\left[\mathrm{e}^{\mathrm{i}(2k\pi)}\right]^{\frac{1}{3}} \\ &= \sqrt[3]{8}\mathrm{e}^{\frac{\mathrm{i}2k\pi}{3}} \qquad \text{using De Moivre's theorem} \end{aligned}$$

Again k can take any integer value or zero. Any three consecutive values will give the roots.

$$\begin{aligned} \text{So if} \quad k &= 0 \qquad z_0 = 2\mathrm{e}^{\mathrm{i}0} = 2 \\ k &= 1 \qquad z_1 = 2\mathrm{e}^{\frac{\mathrm{i}2\pi}{3}} = -1 + \mathrm{i}\sqrt{3} \\ k &= 2 \qquad z_2 = 2\mathrm{e}^{\frac{\mathrm{i}4\pi}{3}} = -1 - \mathrm{i}\sqrt{3} \end{aligned}$$

These are the three (complex) values of $8^{\frac{1}{3}}$ obtained using the exponential form. Of course at the end of the calculation we have converted back to standard Cartesian form.

Following the procedure outlined in the previous example obtain the two complex values of $(1+\mathrm{i})^{1/2}$.

Begin by obtaining the polar form (using the general form of the argument) of $(1+\text{i})$.

Your solution

You should obtain $1+\text{i}=\sqrt{2}(\cos(\frac{\pi}{4}+2k\pi)+\text{i}\sin(\frac{\pi}{4}+2k\pi))\quad k=0,\pm1,\pm2,\ldots$

Now take the square root and use De Moivre's theorem to complete the solution.

Your solution

You should obtain

$$z_1=\sqrt[4]{2}(\cos\frac{\pi}{8}+\text{i}\sin\frac{\pi}{8})=1.099+0.455\text{i}$$

$$z_2=\sqrt[4]{2}(\cos(\frac{\pi}{8}+\pi)+\text{i}\sin(\frac{\pi}{8}+\pi))=-1.099-0.455\text{i}$$

A good exercise would be to repeat the calculation using the exponential form.

Exercises

1. Use De Moivre's theorem to obtain expansions for $\cos 2\theta$ and $\sin 2\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$.
2. Without using tables or a calculator find an expression for $\cos 30^0$ given only that $\cos 90^0=0$.
3. Find all those values of z which satisfy $z^4+1=0$. Write your values in standard Cartesian form.

Answers 1. $\cos 2\theta=2\cos^2\theta-1$ and $\sin 2\theta=2\cos\theta\sin\theta$

2. $\cos 30^0=\dfrac{\sqrt{3}}{2}$

3. $z_0=\dfrac{1}{\sqrt{2}}+\dfrac{\text{i}}{\sqrt{2}}\qquad z_1=-\dfrac{1}{\sqrt{2}}+\dfrac{\text{i}}{\sqrt{2}}\qquad z_2=-\dfrac{1}{\sqrt{2}}-\dfrac{\text{i}}{\sqrt{2}}\qquad z_3=\dfrac{1}{\sqrt{2}}-\dfrac{\text{i}}{\sqrt{2}}$