

Higher Derivatives

11.3



Introduction

The derivative, $\frac{dy}{dx}$, is more expressly called the **first derivative** of y . By differentiating the first derivative, we obtain the **second derivative**; by differentiating the second derivative we obtain the **third derivative** and so on. These second and subsequent derivatives are known as **higher derivatives**.



Prerequisites

Before starting this Section you should ...

- be able to differentiate standard functions



Learning Outcomes

After completing this Section you should be able to ...

- ✓ calculate second and other higher derivatives

1. The derivative of a derivative

You have already learnt how to calculate the derivative of a function using a table of derivatives. By differentiating the function, $y(x)$, we obtain the derivative, $\frac{dy}{dx}$. By repeating the process we can obtain higher derivatives.

Example Calculate the first, second and third derivatives of $y = x^4 + 6x^2$.

Solution

The first derivative is $\frac{dy}{dx}$.

$$\text{first derivative} \equiv \frac{dy}{dx} = 4x^3 + 12x$$

To obtain the second derivative we differentiate the first derivative.

$$\text{second derivative} = 12x^2 + 12$$

The third derivative is found by differentiating the second derivative.

$$\text{third derivative} = 24x + 0 = 24x$$

2. Notation

Just as there is a notation for the first derivative so there is a similar notation for higher derivatives. Consider the function, $y(x)$. We know that the first derivative is $\frac{dy}{dx}$ or $\frac{d}{dx}(y)$ which is the instruction to differentiate the function $y(x)$. The second derivative is calculated by differentiating the first derivative, that is

$$\text{second derivative} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So, using a fairly obvious adaptation of our derivative notation, the second derivative is denoted by $\frac{d^2y}{dx^2}$ and is read as ‘dee two y by dee x squared’. This is often written more concisely as y'' .

In similar manner, the third derivative is denoted by $\frac{d^3y}{dx^3}$ or y''' and so on. So, referring to the Example above we could have written

$$\begin{aligned} \text{first derivative} &= \frac{dy}{dx} = 4x^3 + 12x \\ \text{second derivative} &= \frac{d^2y}{dx^2} = 12x^2 + 12 \\ \text{third derivative} &= \frac{d^3y}{dx^3} = 24x \end{aligned}$$



Key Point

If $y = y(x)$ then its first, second and third derivatives are, denoted respectively by:

$$\frac{dy}{dx} \quad \frac{d^2y}{dx^2} \quad \frac{d^3y}{dx^3}$$

In most examples we use x to denote the independent variable and y the dependent variable. However, in many applications, the time t is the independent variable. In this case a special notation is used for derivatives. Derivatives with respect to t are often indicated using a dot notation, so $\frac{dy}{dt}$ can be written as \dot{y} , pronounced ‘ y dot’. Similarly, a second derivative with respect to t can be written as \ddot{y} , pronounced ‘ y double dot’.



Key Point

$$\dot{y} \text{ stands for } \frac{dy}{dt}, \quad \ddot{y} \text{ stands for } \frac{d^2y}{dt^2} \text{ etc}$$



Calculate $\frac{d^2y}{dt^2}$ and $\frac{d^3y}{dt^3}$ given $y = e^{2t} + \cos t$.

Since $\frac{dy}{dt} = 2e^{2t} - \sin t$, you can now obtain higher derivatives:

Your solution

$$\frac{d^2y}{dt^2} =$$

$$4e^{2t} - \cos t$$

Your solution

$$\frac{d^3y}{dt^3} = \frac{d}{dt} \left(\frac{d^2y}{dt^2} \right) = \frac{d}{dt} (4e^{2t} - \cos t) =$$

$$8e^{2t} + \sin t$$

We could have used the dot notation and written $\dot{y} = 2e^{2t} - \sin t$, and $\ddot{y} = 4e^{2t} - \cos t$.

We may need to evaluate higher derivatives at specific points. We use an obvious notation. The second derivative of $y(x)$, evaluated at say, $x = 2$, is written as $\frac{d^2y}{dx^2}(2)$, or more simply as $y''(2)$.

The third derivative evaluated at $x = -1$ is written as $\frac{d^3y}{dx^3}(-1)$ or $y'''(-1)$.



Given $y(x) = 2 \sin x + 3x^2$ find (a) $y'(1)$ (b) $y''(-1)$ (c) $y'''(0)$

We have $y = 2 \sin x + 3x^2$ and $y' = 2 \cos x + 6x$

Your solution

$$y'' = \frac{d}{dx}(2 \cos x + 6x) = \quad y''' =$$

$$x \cos 2 - = \sin h \quad 9 + x \sin 2 - = \sin h$$

Your solution

(a) $y'(1) =$

(a) $y'(1) = 2 \cos 1 + 6(1) = 7.0806$. Remember, in $\cos 1$ the '1' is 1 radian

Your solution

(b) $y''(-1) =$

$$-2 \sin(-1) + 6 = 7.6829$$

Your solution

(c) $y'''(0) =$

$$y'''(0) = -2 \cos 0 = -2$$

Exercises

1. Find $\frac{d^2y}{dx^2}$ where $y(x)$ is defined by:

(a) $3x^2 - e^{2x}$ (b) $\sin 3x + \cos x$ (c) \sqrt{x} (d) $e^x + e^{-x}$ (e) $1 + x + x^2 + \ln x$

2. Find $\frac{d^3y}{dx^3}$ where y is given in Q1.

3. Calculate $\ddot{y}(1)$ where $y(t)$ is given by:

(a) $t(t^2 + 1)$ (b) $\sin(-2t)$ (c) $2e^t + e^{2t}$ (d) $\frac{1}{t}$ (e) $\cos \frac{t}{2}$

4. Calculate $\ddot{y}(-1)$ of the functions given in Q3.

Answers

1. (a) $6 - 4e^{2x}$ (b) $-9 \sin 3x - \cos x$ (c) $-\frac{1}{x^2} - \frac{1}{3x^{3/2}}$ (d) $e^x + e^{-x}$ (e) $2 - \frac{1}{x^2}$

2. (a) $-8e^{2x}$ (b) $-27 \cos 3x + \sin x$ (c) $\frac{8}{3}x^{-5/2}$ (d) $e^x - e^{-x}$ (e) $\frac{x}{2}$

3. (a) 6 (b) 3.6372 (c) 34.9927 (d) 2 (e) -0.2194

4. (a) 6 (b) -3.3292 (c) 1.8184 (d) -6 (e) -0.0599