Differentiating Products and Quotients 11.4

Introduction

We have seen, in the first three Sections, how standard functions like x^n, e^{ax}, $\sin ax$, $\cos ax$, $\ln ax$ etc., may be differentiated.
In this Section we see how more complicated functions may be differentiated. We concentrate, for the moment, on products and quotients of standard functions like $x^n e^{ax}$, $\frac{e^{ax}\ln x}{\sin x}$ etc.
We will see that two simple rules may be consistently employed to obtain the derivatives of such functions.

Prerequisites

Before starting this Section you should . . .

① be able to differentiate standard functions: logarithms, polynomials, exponentials, and trigonometric functions

② be able to manipulate algebraic expressions.

Learning Outcomes

After completing this Section you should be able to . . .

✓ differentiate products and quotients of standard functions

✓ differentiate a quotient using the product rule

1. Differentiating a Product

In previous Sections we have examined the process of differentiating functions. We found how to obtain the derivative of many commonly occurring functions. These are recorded in the following table (remember, arguments of trigonometric functions are assumed to be in radians).

y	$\dfrac{dy}{dx}$
x^n	nx^{n-1}
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$
$\tan ax$	$a\sec^2 ax$
$\sec ax$	$a\sec x\tan x$
$\ln ax$	$\dfrac{1}{x}$
e^{ax}	ae^{ax}
$\cosh ax$	$a\sinh ax$
$\sinh ax$	$a\cosh ax$

In this Section we consider how to differentiate non-standard functions - in particular those which can be written as the **product** of standard functions. Being able to differentiate such functions depends upon the following Key Point.

Key Point

If $y = f(x)g(x)$ then $\dfrac{dy}{dx} = \dfrac{df}{dx}g(x) + f(x)\dfrac{dg}{dx}$ or $y' = f'g + fg'$

Equivalently, if $y = u.v$ then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

Either of these versions is called the **product rule**.

We shall not prove this result, instead we shall concentrate on its use.

Example Differentiate (a) $y = x^2\sin x$ (b) $y = x\ln x$

Solution

1. (a) Here $f(x) = x^2$, $g(x) = \sin x$ $\quad \therefore \quad \dfrac{\mathrm{d}f}{\mathrm{d}x} = 2x\ , \quad \dfrac{\mathrm{d}g}{\mathrm{d}x} = \cos x$

and so $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x(\sin x) + x^2(\cos x) = x(2\sin x + x\cos x)$

(b) Here $f(x) = x$, $g(x) = \ln x$ $\quad \therefore \quad \dfrac{\mathrm{d}f}{\mathrm{d}x} = 1\ , \quad \dfrac{\mathrm{d}g}{\mathrm{d}x} = \dfrac{1}{x}$

and so $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1.(\ln x) + x.\left(\dfrac{1}{x}\right) = \ln x + 1$

Determine the derivatives of the following functions

(a) $y = e^x \ln x$, $\qquad$ (b) $y = \dfrac{e^{2x}}{x^2}$

Your solution

(a) $f(x) =$ $\qquad \dfrac{\mathrm{d}f}{\mathrm{d}x} =$

$\therefore \quad \dfrac{\mathrm{d}y}{\mathrm{d}x} =$

$g(x) =$ $\qquad \dfrac{\mathrm{d}g}{\mathrm{d}x} =$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^x \ln x + \frac{e^x}{x}$$

In (b), write $y = (x^{-2})e^{2x}$ and then differentiate

Your solution

(b) $f(x) =$ $\qquad \dfrac{\mathrm{d}f}{\mathrm{d}x} =$

$\therefore \quad \dfrac{\mathrm{d}y}{\mathrm{d}x} =$

$g(x) =$ $\qquad \dfrac{\mathrm{d}g}{\mathrm{d}x} =$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (-2x^{-3})e^{2x} + x^{-2}(2e^{2x}) = \frac{2e^{2x}}{x^3}(-1 + x)$$

Exercises

1. In each case find the derivative of the function

   (a) $y = x\tan x$

   (b) $y = x^4\ln(2x)$

   (c) $y = \sin^2 x$

   (d) $y = e^{2x}\cos 3x$

2. Find the derivatives of:

   (a) $y = \dfrac{x}{\cos x}$

   (b) $y = e^x \sin x$

   Obtain the derivative of $y = xe^x \tan x$ using the results of parts (a) and (b).

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1. (a) $\frac{dy}{dx} = \tan x + x\sec^2 x$

   (b) $\frac{dy}{dx} = 4x^3\ln(2x) + \frac{x^4}{x} = x^3(4\ln(2x) + 1)$

   (c) $y = \sin x . \sin x$

   $\therefore \quad \frac{dy}{dx} = \cos x \sin x + \sin x \cos x = 2\sin x\cos x = \sin 2x$

   (d) $\frac{dy}{dx} = (2e^{2x})\cos 3x + e^{2x}(-3\sin 3x) = e^{2x}(2\cos 3x - 3\sin 3x)$

2. (a) $y = x\sec x \qquad \therefore \quad \frac{dy}{dx} = \sec x + x\sec x\tan x$

   (b) $\frac{dy}{dx} = e^x\sin x + e^x\cos x = e^x(\sin x + \cos x)$

   Therefore the derivative of $y = xe^x\tan x = (x\sec x)(e^x\sin x)$ is

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(x\sec x).e^x\sin x + (x\sec x)\frac{d}{dx}(e^x\sin x) \\
&= (\sec x + x\sec x\tan x)e^x\sin x + x\sec x(e^x)(\sin x + \cos x) \\
&= e^x\left\{x + \tan x + x\tan x + x\tan^2 x\right\}
\end{aligned}$$

The rule for differentiating a product can be extended to any number of products. If, for example, $y = f(x)g(x)h(x)$ then

$$\begin{aligned}
\frac{dy}{dx} &= \frac{df}{dx}[g(x)h(x)] + f(x)\frac{d}{dx}[g(x)h(x)] \\
&= \frac{df}{dx}g(x)h(x) + f(x)\left\{\frac{dg}{dx}h(x) + g(x)\frac{dh}{dx}\right\} \\
&= \frac{df}{dx}g(x)h(x) + f(x)\frac{dg}{dx}h(x) + f(x)g(x)\frac{dh}{dx}
\end{aligned}$$

That is, each function in the product is differentiated in turn and the results added together.

Example If $y = xe^{2x}\sin x$ then find $\dfrac{dy}{dx}$

Solution

Here $f(x) = x$, $g(x) = e^{2x}$, $h(x) = \sin x$

$$\frac{df}{dx} = 1, \quad \frac{dg}{dx} = 2e^{2x}, \quad \frac{dh}{dx} = \cos x$$

$$\therefore \quad \frac{dy}{dx} = 1(e^{2x}\sin x) + x(2e^{2x})\sin x + xe^{2x}(\cos x)$$
$$= e^{2x}[\sin x + 2x\sin x + x\cos x]$$

Obtain the first and second derivative of $y = x^2(\ln x)\sinh x$.
Firstly identify your three functions.

Your solution

$f(x) =$ $\qquad g(x) =$ $\qquad h(x) =$

$f(x) = x^2, \quad g(x) = \ln x, \quad h(x) = \sinh x$

Now find the derivative of each of these functions

Your solution

$$\frac{df}{dx} = \qquad \frac{dg}{dx} = \qquad \frac{dh}{dx} =$$

$$\frac{df}{dx} = 2x, \quad \frac{dg}{dx} = \frac{1}{x}, \quad \frac{dh}{dx} = \cosh x$$

Finally obtain $\frac{dy}{dx}$

Your solution

$$\frac{dy}{dx} = 2x(\ln x)\sinh x + x^2\left(\frac{1}{x}\right)\sinh x + x^2\ln x(\cosh x)$$
$$= 2x\ln x\sinh x + x\sinh x + x^2\ln x\cosh x$$

Now find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ by differentiating each of the three terms: $2x\ln x\sinh x,\ x\sinh x,\ x^2\ln x\cosh x$ using the differentiating a product rule. Finally, simplify your answer by collecting like terms together.

Your solution

$$\frac{\mathrm{d}}{\mathrm{d}x}(2x\ln x\sinh x) =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x\sinh x) =$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2\ln x\cosh x) =$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (2+x^2)\ln x\sinh x + 3\sinh x + 2x\cosh x + 4x\ln x\cosh x$$

2. Differentiating Quotients

In this Section we consider functions of the form $y = \dfrac{f(x)}{g(x)}$. To find the derivative of such a function we make use of the following key result:

Key Point

If $y = \dfrac{f(x)}{g(x)}$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{g(x)\dfrac{\mathrm{d}f}{\mathrm{d}x} - \dfrac{\mathrm{d}g}{\mathrm{d}x}f(x)}{[g(x)]^2}$

This is called the **quotient rule**

Example Find the derivative of $y = \dfrac{\ln x}{x}$

Solution

Here $f(x) = \ln x$ and $g(x) = x$

$$\therefore \qquad \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{1}{x} \quad \text{and} \quad \frac{\mathrm{d}g}{\mathrm{d}x} = 1$$

Hence $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x\left(\dfrac{1}{x}\right) - 1(\ln x)}{[x]^2} = \dfrac{1 - \ln x}{x^2}$

Using (a) the formula for differentiating a product and (b) the formula for differentiating a quotient obtain the derivative of $y = \dfrac{\sin x}{x^2}$.

For (a), write $y = x^{-2}\sin x$ then find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Your solution

$y = x^{-2}\sin x \qquad \therefore \qquad \dfrac{\mathrm{d}y}{\mathrm{d}x} = (-2x^{-3})\sin x + x^{-2}\cos x \qquad \therefore \qquad \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{-2\sin x + x\cos x}{x^3}$

Now use the quotient rule to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

Your solution

$y = \dfrac{\sin x}{x^2} \qquad \therefore \qquad \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x^2[\cos x] - (2x)\sin x}{[x^2]^2} = \dfrac{x\cos x - 2\sin x}{x^3}$

Exercises

1. Find the derivatives of the following functions.

   (a) $(2x^3 - 4x^2)(3x^5 + x^2)$

   (b) $\frac{2x^3+4}{x^2-4x+1}$

   (c) $\frac{x^2+2x+1}{x^2-2x+1}$

   (d) $(x^2 + 3)(2x - 5)(3x + 2)$

   (e) $\frac{(2x+1)(3x-1)}{x+5}$

   (f) $(\ln x)\sin x$

   (g) $(\ln x)/\sin x$

   (h) e^x/x^2

   (i) $\frac{e^x \sin x}{\cos 2x}$

Answers

1. (a) $48x^7 - 84x^6 + 10x^4 - 16x^3$

   (b) $\frac{2x^4-16x^3+6x^2-8x+16}{(x^2-4x+1)^2}$

   (c) $-\frac{4(x+1)}{(x-1)^3}$

   (d) $24x^3 - 33x^2 + 16x - 33$

   (e) $\frac{6(x^2+10x+1)}{(x+5)^2}$

   (f) $\frac{1}{x}\sin x + (\ln x)\cos x$

   (g) $\frac{\sin x\left(\frac{1}{x}\right)-(\ln x)\cos x}{\sin^2 x} = \mathrm{cosec}\, x\left(\frac{1}{x} - \cot x \ln x\right)$

   (h) $\frac{x^2e^x-2xe^x}{x^4} = (x^{-2} - 2x^{-3})e^x$

   (i) $\frac{\cos 2x[e^x \sin x + e^x \cos x] + 2\sin 2x e^x \sin x}{\cos^2 2x} = e^x[(\sin x + \cos x)\sec 2x + 2\sin x \sin 2x \sec^2 2x]$