Parametric Differentiation

11.6

Introduction

Often, the equation of a curve may not be given in Cartesian form $y = f(x)$ but in parametric form: $x = h(t),\ y = g(t)$. In this section we see how to calculate the derivative $\frac{dy}{dx}$ from a knowledge of the so-called parametric derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$. We then extend this to the determination of the second derivative $\frac{d^2y}{dx^2}$.

Parametric functions arise often in dynamics in which the parameter t represents the time and $(x(t),\ y(t))$ then represents the position of a particle as it varies with time.

Prerequisites

Before starting this Section you should ...

① be able to differentiate standard functions

② be able to plot a curve given in parametric form

Learning Outcomes

After completing this Section you should be able to ...

✓ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when the equation of a curve is given in parametric form.

1. Parametric Differentiation

In this section we consider the parametric approach to describing a curve:

$$\underbrace{x = h(t) \qquad y = g(t)}_{\text{parametric equations}} \qquad\qquad \underbrace{t_0 \leq t \leq t_1}_{\text{parametric range}}$$

As various values of t are chosen within the parameter range the corresponding values of x, y are calculated from the parametric equations. When these points are plotted on an xy plane they trace out a curve. The Cartesian equation of this curve is obtained by eliminating the parameter t from the parmatric equations. For example, consider the curve:

$$x = 2\cos t \qquad y = 2\sin t \qquad 0 \leq t \leq 2\pi.$$

We can eliminate the $t-$variable in an obvious way (divide both parametric equations by 2, square each and then add):

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = \cos^2 t + \sin^2 t = 1$$

$$\therefore \qquad x^2 + y^2 = 4$$

which we recognise as the standard equation of a circle with centre at (0,0) with radius 2.
In a similar fashion the parametric equations

$$x = 2t \qquad y = 4t^2 \qquad -\infty < t < \infty$$

describes a parabola. This follows since, eliminating the parameter t:

$$t = \frac{x}{2} \qquad \therefore \qquad y = 4\left(\frac{x^2}{4}\right) = x^2$$

which we recognise as the standard equation of a parabola.
The question we wish to address in this section is 'how do we obtain the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ if a curve is given in parametric form?' To answer this we note the key result in this area:

Key Point

If $x = h(t)$ and $y = g(t)$ then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t}$$

We note that this result allows the determination of $\frac{\mathrm{d}y}{\mathrm{d}x}$ without the need to find y as an explicit function of x.

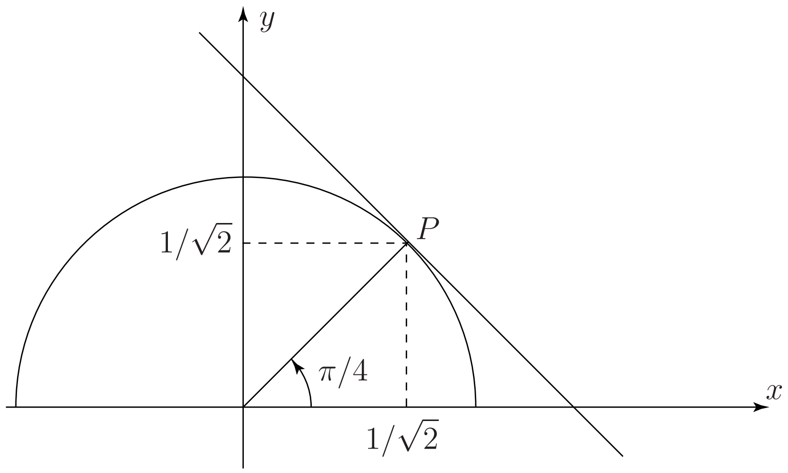
Example Determine the equations of the tangent line to the semi-circle

$$x = \cos t \qquad y = \sin t \qquad 0 \leq t \leq \pi$$

at $t = \pi/4$

Solution

The semi-circle is drawn in the figure



We have also drawn the tangent line at $t = \pi/4$ (or, equivalently, at $x = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $y = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.) Now

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\cos t}{-\sin t} = -\cot t.$$

Thus at $t = \frac{\pi}{4}$ we have $\frac{\mathrm{d}y}{\mathrm{d}x} = -\cot\left(\frac{\pi}{4}\right) = -1$. The equation of the tangent line is

$$y = mx + c$$

where m is the gradient of the line and c is a constant.

Clearly $m = -1$ (since, at the point P the line and the circle have the same gradient).

To find c we note that the line passes through the point P with coordinates $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Hence

$$\frac{1}{\sqrt{2}} = (-1)\frac{1}{\sqrt{2}} + c \qquad \therefore \qquad c = \frac{2}{\sqrt{2}}$$

Finally,

$$y = -x + \frac{2}{\sqrt{2}}$$

is the equation of the tangent line at the point in question.

We should note, before proceeding, that a derivative with respect to the parameter t is often denoted by a 'dot'. Thus

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x}, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \dot{y}, \qquad \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \ddot{x} \ \text{ etc.}$$

Find the value of $\frac{dy}{dx}$ if

$$x = 3t, \quad y = t^2 - 4t + 1$$

Check your result by finding $\frac{dy}{dx}$ in the normal way.

First find $\dfrac{dx}{dt}, \dfrac{dy}{dt}$

Your solution

$$\frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 2t - 4$$

Now obtain $\dfrac{dy}{dx}$

Your solution

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{2t-4}{3} = \frac{2}{3}t - \frac{4}{3}, \text{ or, using the 'dot' notation } \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{2t-4}{3} = \frac{2}{3}t - \frac{4}{3}$$

Now find y explicitly as a function of x. Then, find $\dfrac{dy}{dx}$ directly.

Your solution

$$t = \frac{x}{3} \quad \therefore \quad y = \frac{x^2}{9} - \frac{4x}{3} + 1. \quad \text{Finally: } \frac{dy}{dx} = \frac{2x}{9} - \frac{4}{3} = \frac{2t}{3} - \frac{4}{3}.$$

Find the value of $\frac{dy}{dx}$ at $t = 2$ if

$$x = 3t - 4\sin\pi t \qquad y = t^2 + t\cos\pi t \qquad 0 \le t \le 4$$

First find $\dfrac{dx}{dt}, \dfrac{dy}{dt}$

Your solution

$$\frac{dx}{dt} = 3 - 4\pi\cos\pi t \qquad \frac{dy}{dt} = 2t + \cos\pi t - \pi t\sin\pi t$$

Now obtain $\dfrac{dy}{dx}$

Your solution

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt} = \dfrac{2t+\cos \pi t - \pi t \sin \pi t}{3 - 4\pi \cos \pi t}$ or, using the dot notation, $\dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}} = \dfrac{2t+\cos \pi t - \pi t \sin \pi t}{3 - 4\pi \cos \pi t}$

Finally, substitute $t = 2$ to find $\frac{dy}{dx}$ at this value of t.

Your solution

$\dfrac{dy}{dx}\Big|_{t=2} = \dfrac{4+1}{3-4\pi} = \dfrac{5}{3-4\pi} = -0.523$

2. Higher Derivatives

Having found the derivative $\frac{dy}{dx}$ using parametric differentiation we now ask how we might determine the second derivative $\frac{d^2y}{dx^2}$.
By definition:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

But

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \qquad \text{and so} \qquad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\dot{y}}{\dot{x}}\right)$$

Now $\frac{\dot{y}}{\dot{x}}$ is a function of t so we can change the derivative with respect to x into a derivative with respect to t since

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \left\{\frac{d}{dt}\left(\frac{dy}{dx}\right)\right\}\frac{dt}{dx}$$

from the function of a function rule (see section 12.2).
But, differentiating the quotient $\dot{y}/\dot{x}$ we have

$$\frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right) = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2}$$

and

$$\frac{dt}{dx} = \frac{1}{\left(\frac{dx}{dt}\right)} = \frac{1}{\dot{x}}$$

so finally:

$$\frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$$

Key Point

If $x = h(t), \quad y = g(t)$ then the first and second derivatives of y with respect to x are:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} \qquad \text{and} \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$$

Example If the parametric equations of a curve are

$$x = 2t, \quad y = t^2 - 3, \quad -4 < t < 4$$

determine $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

Solution

Here $\dot{x} = 2, \ \dot{y} = 2t$

$$\therefore \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{2t}{2} = t.$$

Also $\ddot{x} = 0, \quad \ddot{y} = 2$

$$\therefore \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{2(2) - 2t(0)}{(2)^3} = \frac{1}{2}$$

These results can easily be checked in this case since $t = \frac{x}{2}$ and $y = t^2 - 3$ which imply $y = \frac{x^2}{4} - 3$. Therefore the derivatives can be obtained directly:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{4} = \frac{x}{2} \qquad \text{and} \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{2}.$$

Exercises

1. For the following sets of parametric equations find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

   (a) $x = 3t^2 \qquad y = 4t^3$

   (b) $x = 4 - t^2 \qquad y = t^2 + 4t$

   (c) $x = t^2 e^t \qquad y = t$

2. Find the equation of the tangent line to the curve:

$$x = 1 + 3\sin t \qquad y = 2 - 5\cos t \qquad \text{at} \quad t = \frac{\pi}{6}$$

Answers

1. (a) $\frac{dy}{dx} = 2t, \quad \frac{d^2y}{dx^2} = \frac{1}{3t}$.

   (b) $\frac{dy}{dx} = -1 - \frac{2}{t}, \quad \frac{d^2y}{dx^2} = -\frac{1}{t^3}$

   (c) $\frac{dy}{dx} = \frac{e^{-t}}{2t+t^2}, \quad \frac{d^2y}{dx^2} = -\frac{e^{-2t}(t^2+4t+2)}{(t+2)^3t^3}$

2. $\dot{x} = 3\cos t \qquad \dot{y} = +5\sin t$

$$\therefore \quad \frac{dy}{dx} = \frac{5}{3}\tan t \qquad \therefore \quad \left.\frac{dy}{dx}\right|_{t=\pi/6} = \frac{5}{3}\tan\frac{\pi}{6} = \frac{5}{3}\frac{1}{\sqrt{3}} = \frac{5\sqrt{3}}{9}$$

The equation of the tangent line is $y = mx + c$ where $m = \frac{5\sqrt{3}}{9}$.

Now the line passes through the point $x = 1 + 3\sin\frac{\pi}{6} = 1 + \frac{3}{2}$, $y = 2 - 5\frac{\sqrt{3}}{2}$ and so

$$2 - 5\frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{9}(1 + \frac{3}{2}) + c \qquad \therefore \quad c = 2 - \frac{35\sqrt{3}}{9}$$