Implicit Differentiation

11.7

Introduction

This Section introduces implicit differentiation which is used to differentiate functions expressed in implicit form (where the variables are found together). Examples are $x^2 + xy + y^2 = 1$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which represents an ellipse.

Prerequisites

Before starting this Section you should ...

① be able to differentiate standard functions

② be familiar with the chain rule

Learning Outcomes

After completing this Section you should be able to ...

✓ able to differentiate function expressed implicitly

1. Implicit and Explicit Functions

Equations such as $y = x^2$, $y = \dfrac{1}{x}$, $y = \sin x$ are said to define y **explicitly** as a function of x because the variable y appears alone on one side of the equation.

The equation

$$yx + y + 1 = x$$

is not of the form $y = f(x)$ but can be put into this form by simple algebra.

Write y as the subject of

$$yx + y + 1 = x \qquad (1)$$

Your solution

We have $y(x+1) = x - 1$ so

$$y = \frac{x-1}{x+1}$$

We say that by means of (1) y is defined **implicitly** as a function of x, the actual function being given as

$$y = \frac{x-1}{x+1} \qquad (2)$$

We should note than an equation relating x and y can implicitly define **more than one** function of x.

For example if we solve

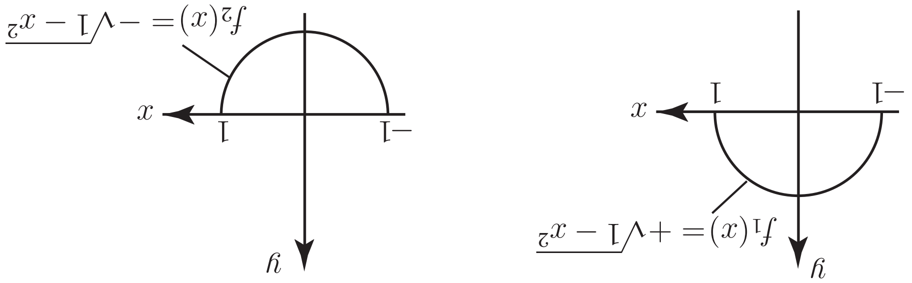
$$x^2 + y^2 = 1 \qquad (3)$$

we obtain $y = \pm\sqrt{1-x^2}$ so (3) defines implicitly two functions

$$f_1(x) = \sqrt{1-x^2} \qquad f_2(x) = -\sqrt{1-x^2}$$

Sketch the graphs of these two functions (Equation (3) should give you the clue.)

Your solution



Since $x^2 + y^2 = 1$ is the well-known equation of the circle centre at the origin and radius 1 it follows that the graphs of $f_1(x)$ and $f_2(x)$ are respectively the upper and lower halves of this circle.

Sometimes it is difficult or even impossible to solve an equation in x and y to obtain y explicitly in terms of x.

Examples where explicit expressions for y cannot be obtained are

$$\sin(xy) = y \qquad x^2 + \sin y = 2y$$

2. Differentiation of Implicit Functions

It is not necessary fortunately to have to solve an equation to obtain y in terms of x in order to **differentiate** a function defined implicitly.

Consider firstly the simple equation

$$xy = 1$$

Here it **is** clearly possible to obtain y as the subject of this equation and hence obtain the derivative $\dfrac{dy}{dx}$.

Express y explicitly in terms of x and find $\dfrac{dy}{dx}$ for the case $xy = 1$.

Your solution

We have immediately

$$y = \frac{1}{x} \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x^2}$$

We now show an alternative way of obtaining $\frac{\mathrm{d}y}{\mathrm{d}x}$ if $xy = 1$ which does **not** involve writing y explicitly in terms of x at the outset. We simply treat y as an (unspecified) function of x.

Hence if $\quad xy = 1 \quad$ we obtain

$$\frac{\mathrm{d}(xy)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(1).$$

The right-hand side differentiates to zero as 1 is a constant. On the left-hand side we must use the **product** rule of differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x\frac{\mathrm{d}y}{\mathrm{d}x} + y\frac{\mathrm{d}x}{\mathrm{d}x} = x\frac{\mathrm{d}y}{\mathrm{d}x} + y$$

Hence $xy = 1$ becomes, after differentiation,

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0 \quad \text{or} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x}$$

In this case we can of course substitute $y = \frac{1}{x}$ to obtain

$$y = -\frac{1}{x^2}$$

as before.

The method used here is called **implicit differentiation** and, apart from the final step, it can be applied even if y cannot be expressed explicitly in terms of x. Indeed, on occasions, it is **easier** to differentiate implicitly even if an explicit expression is possible.

Obtain the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ if $\qquad x^2 + y = 1 + y^3. \qquad$ (4)

Begin by differentiating the left-hand side with respect to x.

Your solution

We obtain, for the left-hand side,

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + y) = 2x + \frac{\mathrm{d}y}{\mathrm{d}x}$$

Now differentiate the right hand side of (4) with respect to x. You will need to use the chain (or function of a function) rule to deal with the term y^3.

Your solution

We obtain for the right-hand side

$$\frac{\mathrm{d}}{\mathrm{d}x}(1 + y^3) = \frac{\mathrm{d}}{\mathrm{d}x}(1) + \frac{\mathrm{d}}{\mathrm{d}x}(y^3) = 0 + 3y^2\frac{\mathrm{d}y}{\mathrm{d}x}$$

Hence, finally we have, equating the left- and right-hand side derivatives of (4):

$$2x + \frac{\mathrm{d}y}{\mathrm{d}x} = 3y^2\frac{\mathrm{d}y}{\mathrm{d}x}$$

We can make $\frac{\mathrm{d}y}{\mathrm{d}x}$ the subject of this equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} = -2x \quad \text{which gives} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{3y^2 - 1}$$

We note that $\frac{\mathrm{d}y}{\mathrm{d}x}$ has to be expressed in terms of x and y. This is quite usual if y cannot be obtained explicitly in terms of x.

Now try a further example of implicit differentiation.

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ if $\quad 2y = x^2 + \sin y \qquad (5)$

Your answer will be in terms of y and x.

Your solution

We have, on differentiating both sides of (5) with respect to x and using the chain rule on the $\sin y$ term:

$$\frac{\mathrm{d}}{\mathrm{d}x}(2y) = \frac{\mathrm{d}}{\mathrm{d}x}(x^2) + \frac{\mathrm{d}}{\mathrm{d}x}(\sin y) \longrightarrow 2\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + \cos y \frac{\mathrm{d}y}{\mathrm{d}x}$$

leading to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{2 - \cos y}.$$

We sometimes need to obtain the second derivative $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ for a function defined implicitly.

Example If $$x^2 - xy - y^2 - 2y = 0 \qquad (6)$$

obtain $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at the point (4,2) on the curve defined by the equation.

Solution

Firstly we obtain $\frac{\mathrm{d}y}{\mathrm{d}x}$ from (6) and then evaluate it at (4,2).

We have

$$2x - x\frac{\mathrm{d}y}{\mathrm{d}x} - y - 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 2\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \qquad (7)$$

from which

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x - y}{x + 2y + 2} \qquad (8)$$

so at (4,2) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6}{10} = \frac{3}{5}$.

To obtain the second derivative $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ it is easier to use (7) than (8) because the latter is a quotient. We simplify (7) first:

$$2x - y - (x + 2y + 2)\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \qquad (9)$$

We will have to use the product rule to differentiate the third term here. Hence differentiating (9) with respect to x:

$$2 - \frac{\mathrm{d}y}{\mathrm{d}x} - (x + 2y + 2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - (1 + 2\frac{\mathrm{d}y}{\mathrm{d}x})\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

or

$$2 - 2\frac{\mathrm{d}y}{\mathrm{d}x} - 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - (x + 2y + 2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \qquad (10)$$

Solution

Note carefully that the third term here, $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$, is the square of the first derivative. It should not be confused with the second derivative denoted by $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

Finally, at (4,2) where $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{5}$ we obtain from (10):

$$2 - 2(\frac{3}{5}) - 2(\frac{9}{25}) - (4 + 4 + 2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$$

from which

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{125} \text{ at } (4,2).$$

This exercise involves the curvature of a bent beam. When a horizontal beam is acted on by forces which bend it, then each small segment of the beam will be slightly curved and can be regarded as an arc of a circle. The radius R of that circle is called the **radius of curvature** of the beam at the point concerned. If the shape of the beam is described by an equation of the form $y = f(x)$ then there is a formula for the radius of curvature R which involves only the first and second derivatives $\mathrm{d}y/\mathrm{d}x$ and $\mathrm{d}^2y/\mathrm{d}x^2$.

Find that equation as follows.

Start with the equation of a circle in the simple implicit form

$$x^2 + y^2 = R^2$$

and perform implicit differentiation twice. Now use the result of the first implicit differentiation to find a simple expression for the quantity $1+(\mathrm{d}y/\mathrm{d}x)^2$; this can then be used to simplify the result of the second differentiation.

Your solution

$$x^2 + y^2 = R^2$$

differentiating: $$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \qquad (11)$$

differentiating again: $$2 + 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \qquad (12)$$

where the product rule of differentiation has been used to obtain the second and third terms.

From (11)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$$

$$\therefore \quad 1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1 + \frac{x^2}{y^2} = \frac{y^2 + x^2}{y^2} = \left(\frac{R}{y}\right)^2 \qquad (13)$$

Thus (12) becomes $$2\left(\frac{R}{y}\right)^2 + 2y\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) = 0$$

so $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{R^2}{y^3} = -\left(\frac{1}{R}\right)\left(\frac{R}{y}\right)^3$$

Using (13) then gives the result

$$\frac{1}{R} = -\frac{\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}}{[1 + \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2]^{3/2}}$$

The usual textbook equation omits the minus sign but the sign indicates whether the circle is above or below the curve, as you will see by sketching a few examples. When the gradient is small (as for a slightly deflected horizontal beam), i.e. $\frac{\mathrm{d}y}{\mathrm{d}x}$ is small, the denominator in the equation for $(1/R)$ is close to 1, and so the second derivative alone is often used to estimate the radius of curvature in the theory of bending beams.