Contents 12

applications of **differentiation**

Learning **outcomes**

In this workbook you will learn to apply your knowledge of differentiation to solve some basic problems connected with curves. First you will learn how to obtain the equation of the tangent line and the normal line to any point of interest on a curve. Secondly, you will learn how to find the positions of maxima and minima on a given curve. Thirdly, you will learn how, given an approximate position of the root of a function, a better estimate of the position can be obtained using the Newton-Raphson technique. Lastly you will learn how to characterise how sharply a curve is turning by calculating its curvature.

Time **allocation**

You are expected to spend approximately seven hours of independent study on the material presented in this workbook. However, depending upon your ability to concentrate and on your previous experience with certain mathematical topics this time may vary considerably.

Tangents and Normals 12.1

Introduction

In this section we see how the equations of the tangent line and the normal line at a particular point on the curve $y = f(x)$ can be obtained. The equations of tangent and normal lines have form:

$$y = mx + c, \qquad y = nx + d$$

respectively. We shall show that the product mn is -1 if these lines are to be perpendicular. The constants c, d are then obtained by using the property that the both the normal and tangent lines and the curve pass through a common point.

Prerequisites

Before starting this Section you should . . .

① be able to differentiate standard functions

② understand the geometrical interpretation of a derivative

③ understand the trigonometric expansions of $\sin(A + B)$, $\cos(A + B)$

Learning Outcomes

After completing this Section you should be able to . . .

✓ obtain the condition that two given lines be perpendicular

✓ obtain the equation of the tangent line to a curve

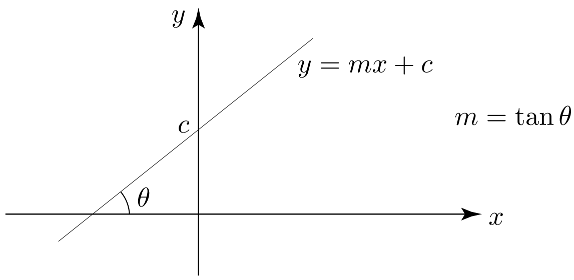
✓ obtain the equation of the normal line to a curve

1. Perpendicular Lines

One form for the equation of a straight line is

$$y = mx + c$$

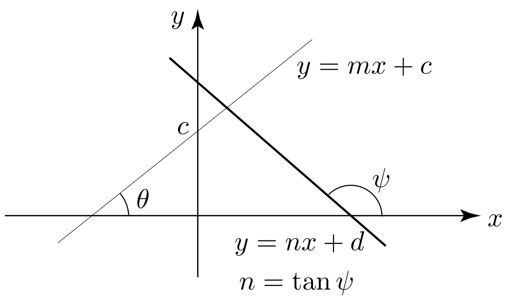
where m and c are constants. We remember that m is the gradient of the line and its value is the tangent of the angle θ that the line makes with the positive $x-$axis. The constant c is the value obtained where the line intersects the $y-$axis. See the following diagram:



If we have a second line, with equation

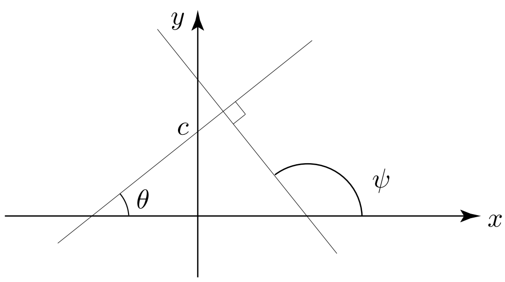
$$y = nx + d$$

then, unless $m = n$, the two lines will intersect. These are drawn together on the following diagram. The second line makes an angle ψ with the positive $x-$axis.



A simple question to ask is "what is the relation between m and n if the lines are perpendicular?" If the lines are perpendicular, as shown in the next figure, the angles θ, ψ must satisfy the relation:

$$\psi - \theta = 90°$$



This is true since the angles in a triangle add up to 180°. According to the figure the three angles are 90°, θ and $180° - \psi$. Therefore

$$180° = 90° + \theta + (180° - \psi) \qquad \text{implying} \qquad \psi - \theta = 90°$$

In this special case that the lines are perpendicular or **normal** to each other the relation between the gradients m and n is easily obtained. In this deduction we use the following basic trigonometric relations and identities:

$$\sin(A - B) = \sin A \cos B - \cos A \sin B \qquad \cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\tan A = \frac{\sin A}{\cos A} \qquad \sin 90^\circ = 1 \qquad \cos 90^\circ = 0$$

Now

$$\begin{aligned} m = \tan\theta &= \tan(\psi - 90^o) \qquad \text{(see previous figure)} \\ &= \frac{\sin(\psi - 90^o)}{\cos(\psi - 90^o)} \\ &= \frac{-\cos\psi}{\sin\psi} = -\frac{1}{\tan\psi} = -\frac{1}{n} \\ \text{So} \qquad mn &= -1 \end{aligned}$$

Key Point

Two straight lines $y = mx + c$, $y = nx + d$ are perpendicular if

$$m = -\frac{1}{n} \qquad \text{or equivalently} \qquad mn = -1$$

This result assumes that neither of the lines are parallel to the $x-$axis or to the $y-$axis, as in such cases one gradient will be zero and the other infinite.

Exercises

Which of the following pairs of lines are perpendicular:

i. $y = -x + 1, \quad y = x + 1$
ii. $y + x - 1 = 0, \quad y + x - 2 = 0$
iii. $2y = 8x + 3, \quad y = -0.25x - 1$

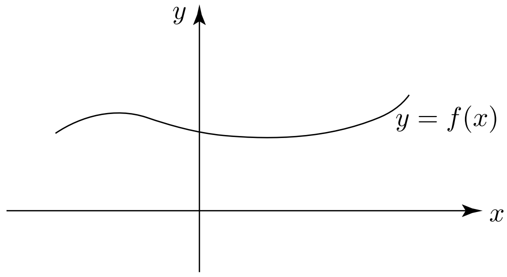
Answers (i) perpendicular (ii) not perpendicular (iii) perpendicular

2. Tangents and Normals to a curve

As we know, the relationship between an independent variable x and a dependent variable y is denoted by

$$y = f(x)$$

As we also know, the geometrical interpretation of this relation takes the form of a curve in an xy plane as illustrated in the following diagram.



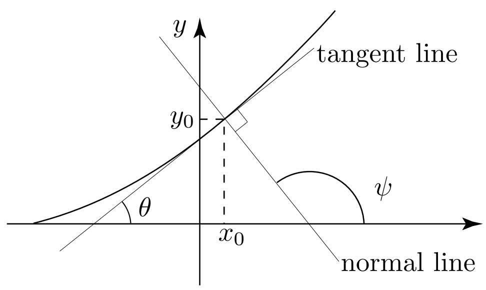
We know how to calculate a value of y given a value of x. We can either do this graphically (which is inaccurate) or else use the function itself. So, at an x value of x_0 the corresponding y value is y_0 where

$$y_0 = f(x_0)$$

Let us examine the curve in the neighbourhood of the point (x_0, y_0). There are two important constructions of interest

- the tangent line at (x_0, y_0)
- the normal line at (x_0, y_0)

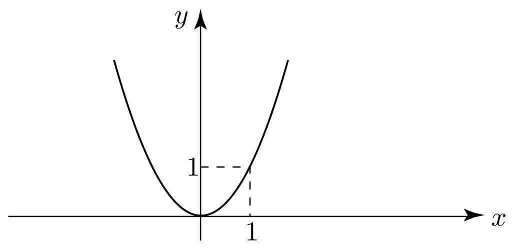
These are shown in the following figure

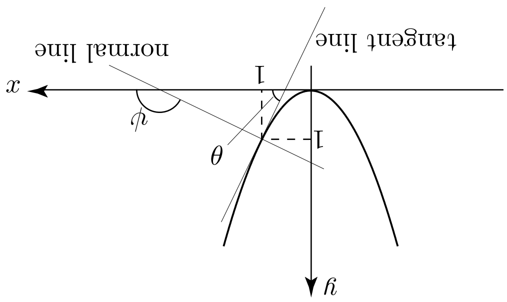


We note the geometrically obvious fact: the tangent and normal lines at any given point on a curve are perpendicular to each other.

The curve $y = x^2$ is drawn below. On this graph draw the tangent line and the normal line at the point $(x_0 = 1, y_0 = 1)$

Your solution





From your graph, estimate the values of θ and ψ. (You will need a protractor).

Your solution

$\theta \simeq$ $\qquad\qquad$ $\psi \simeq$

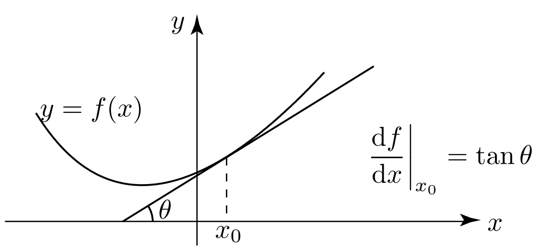
$\theta = 63.4^o$ $\qquad\qquad$ $\psi = 153.4^o$

Returning to the curve $y = f(x)$: we know, from the geometrical interpretation of the derivative that

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_0} = \tan\theta$$

(in case you've forgotten, the notation on the left-hand side of this relation means evaluate $\frac{\mathrm{d}f}{\mathrm{d}x}$ at the value x_0)

Also, in this relation, θ is the angle the tangent line to the curve $y = f(x)$ makes with the positive $x-$axis. This is highlighted in the following diagram:



3. The Tangent Line to a Curve

Let the equation of the tangent line to the curve $y = f(x)$ at the point (x_0, y_0) be:

$$y = mx + c$$

where m and c are constants to be found. The line just touches the curve $y = f(x)$ at the point (x_0, y_0) so, at this point both must have the **same value** for the derivative. That is:

$$m = \left.\frac{df}{dx}\right|_{x_0}$$

Since we know (in any particular case) $f(x)$ and the value x_0 we can readily calculate the value for m. The value of c is found by using the fact that the tangent line and the curve pass through the same point (x_0, y_0).

$$y_0 = mx_0 + c \qquad \text{and} \qquad y_0 = f(x_0)$$

Thus

$$mx_0 + c = f(x_0) \qquad \text{leading to} \qquad c = f(x_0) - mx_0$$

Key Point

The equation of the tangent line to the curve $y = f(x)$ at the point (x_0, y_0) is

$$y = mx + c \quad \text{where} \quad m = \left.\frac{df}{dx}\right|_{x_0} \quad \text{and} \quad c = f(x_0) - mx_0$$

Example Find the equation of the tangent line to the curve $y = x^2$ at the point (1,1)

Solution

Here $f(x) = x^2$ and $x_0 = 1$.

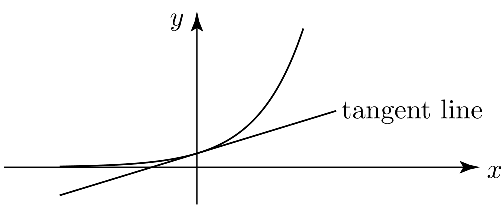
Thus $\frac{df}{dx} = 2x \quad \therefore \quad m = \left.\frac{df}{dx}\right|_{x_0} = 2$

Also $c = f(x_0) - mx_0 = f(1) - m = 1 - 2 = -1$.

Therefore the equation of the tangent line is $y = 2x - 1$.

(Check back to the previous guided exercise to see if this 'looks right').

Find the equation of the tangent line to the curve $y = e^x$ at the point $x = 0$.
The curve and the line are displayed in the following figure:



Specify x_0 and f

Your solution

$x_0 =$ $\qquad\qquad$ $f(x) =$

$x_0 = 0 \qquad f(x) = e^x$

Now obtain the values of $\left.\dfrac{df}{dx}\right|_{x_0}$, $f(x_0) - mx_0$

Your solution

$\left.\dfrac{df}{dx}\right|_{x_0} =$ $\qquad\qquad$ $f(x_0) - mx_0 =$

$\dfrac{df}{dx} = e^x \quad \therefore \quad \left.\dfrac{df}{dx}\right|_0 = 1 \quad$ and $\quad f(0) - 1(0) = e^0 - 0 = 1$

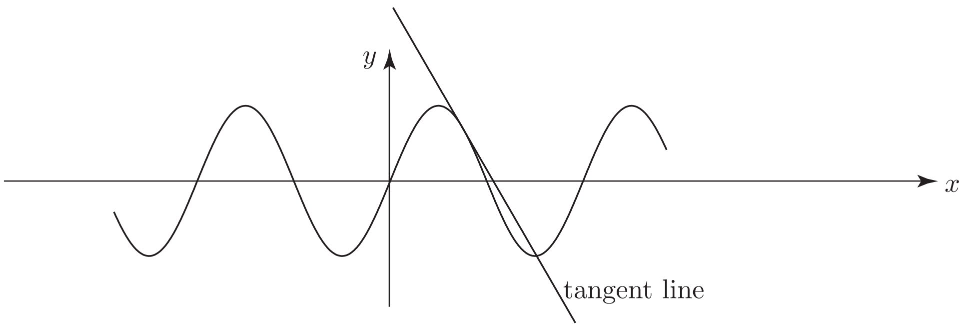
Now obtain the equation of the tangent line

Your solution

$y =$

$y = x + 1$

Find the equation of the tangent line to the curve $y = \sin 3x$ at the point $x = \frac{\pi}{4}$ and find where the tangent line intersects the $x-$axis. See the following figure.



Specify x_0 and f

Your solution

$x_0 =$ $\qquad\qquad$ $f(x) =$

$x_0 = \frac{\pi}{4} \qquad f(x) = \sin 3x$

Now obtain the values of $\left.\dfrac{df}{dx}\right|_{x_0}$, $f(x_0) - mx_0$

Your solution

$\left.\frac{df}{dx}\right|_{x_0} =$ $\qquad\qquad\qquad f(x_0) - mx_0 =$

$\frac{df}{dx} = 3\cos 3x \qquad \therefore \qquad \left.\frac{df}{dx}\right|_{\frac{\pi}{4}} = 3\cos\frac{3\pi}{4} = -\frac{3}{\sqrt{2}} = -2.12$

and $f\left(\frac{\pi}{4}\right) - \frac{m\pi}{4} = \sin\frac{3\pi}{4} - \left(\frac{-3}{\sqrt{2}}\right)\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{3}{\sqrt{2}}\frac{\pi}{4} = 2.37$ to 2 d.p.

Now obtain the equation of the tangent line

Your solution

$y =$

$y = \frac{-3}{\sqrt{2}}x + \frac{1}{4\sqrt{2}}(4 + 3\pi) = -2.12x + 2.37$

where does the line intersect the $x-$axis?

Your solution

$x =$

when $y = 0 \quad \therefore \quad -2.12x + 2.37 = 0 \quad \therefore \quad x = 1.12$ to 2 d.p.

4. The Normal Line to a Curve

We have already noted that, at any point (x_0, y_0) on a curve $y = f(x)$ the tangent and normal lines are perpendicular. Thus if the equations of the tangent and normal lines are, respectively

$$y = mx + c \qquad y = nx + d$$

then $m = -\frac{1}{n}$ or, equivalently $n = -\frac{1}{m}$.
We have also noted, for the tangent line

$$m = \left.\frac{df}{dx}\right|_{x_0}$$

so n can easily be obtained. To find d, we again use the fact that the normal line $y = nx + d$ and the curve have a point in common:

$$y_0 = nx_0 + \mathrm{d} \qquad \text{and} \qquad y_0 = f(x_0)$$

so $nx_0 + \mathrm{d} = f(x_0)$ leading to $\mathrm{d} = f(x_0) - nx_0$.

Find the equation of the normal line to curve $y = \sin 3x$ at the point $x = \frac{\pi}{4}$.
See the earlier guided exercise for the value of 'm'.

Your solution

$$m = \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{\frac{\pi}{4}} =$$

$m = \frac{-3}{\sqrt{2}}$

Hence find the value of n

Your solution

$nm = -1 \qquad \therefore \qquad n =$

$nm = -1 \qquad \therefore \qquad n = \frac{\sqrt{2}}{3}$

The equation of the normal line is $y = \frac{\sqrt{2}}{3}x + \mathrm{d}$.
Now find the value of d. (Remember the normal line and the curve pass through the same point.)

Your solution

$\frac{\sqrt{2}}{3}\left(\frac{\pi}{4}\right) + \mathrm{d} = \sin\frac{\pi}{4} \qquad \therefore \qquad \mathrm{d} = \frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{3}\frac{\pi}{4} \simeq 0.34$

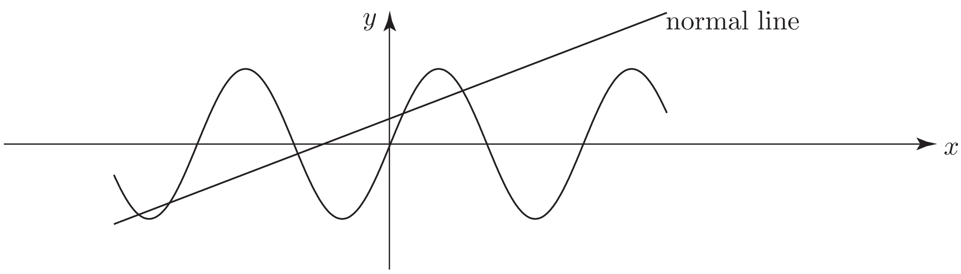
Now obtain the equation of the normal line.

Your solution

$y =$

$y = 0.47x + 0.34$

The curve and the normal line are shown in the following figure:



Find the equation of the normal line to the curve $y = x^3$ at $x = 1$.

First find $f(x),\ x_0,\ \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_0},\ m,\ n$

Your solution

$$f(x) = x^3, \; x_0 = 1, \quad \left.\frac{df}{dx}\right|_1 = 3x^2\Big|_1 = 3$$

$$\therefore \quad m = 3 \text{ and } n = -\tfrac{1}{3}$$

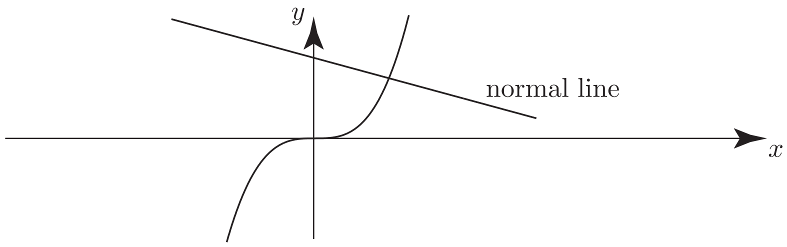
Now use the property that the normal line $y = nx + \mathrm{d}$ and the curve $y = x^3$ pass through the point (1,1)

Your solution

d =

$$1 = n + \mathrm{d} \qquad \therefore \qquad \mathrm{d} = 1 - n = 1 + \tfrac{1}{3} = \tfrac{4}{3}$$

Thus the equation of the normal line is $y = -\frac{1}{3}x + \frac{4}{3}$. The curve and the normal line are shown below:



Exercises

1. Find the equations of the tangent and normal lines to the following curves at the points indicated

   (a) $y = x^4 + 2x^2$ $\qquad (1, 3)$

   (b) $y = \sqrt{1 - x^2}$ $\qquad (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$

   (What would be obtained if the point was $(1, 0)$?)

   (c) $y = x^{\frac{1}{2}}$ $\qquad (1, 1)$

2. Find the value of a $(a \neq -1)$ if the two curves $y = e^{-x}$ and $y = e^{ax}$ are to intersect at right-angles.

Answers

2. The curves will intersect at right-angles if their tangent lines, at the point of intersection, are perpendicular.

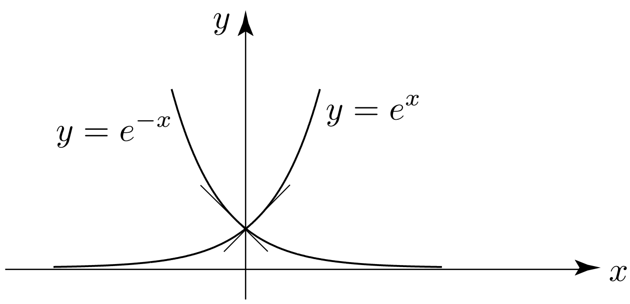
Point of intersection: $e^{-x} = e^{ax}$ i.e. $-x = ax$ $\therefore$ $x = 0$

The tangent line to $y = e^{ax}$ is $y = mx + c$ where $m = ae^{ax}\big|_{x=0} = a$

The tangent line to $y = e^{-x}$ is $y = kx + g$ where $k = -e^{-x}\big|_{x=0} = -1$

These two lines are perpendicular if

$$a(-1) = -1 \qquad \text{i.e.} \qquad a = 1.$$



Answers

1. (a) $f(x) = x^4 + 2x^2$ $\quad \frac{\mathrm{d}f}{\mathrm{d}x} = 4x^3 + 4x,$ $\quad \frac{\mathrm{d}f}{\mathrm{d}x}\Big|_{x=1} = 8$

tangent line $y = 8x + c$. This passes through $(1, 3)$ so

$3 = 8 + c \quad \therefore \quad c = -5 \quad \therefore \quad y = 8x - 5$

normal line $y = -\frac{1}{8}x + \mathrm{d}$. This passes through $(1, 3)$ so

$3 = -\frac{1}{8} + \mathrm{d} \quad \therefore \quad \mathrm{d} = \frac{25}{8} \quad \therefore \quad y = -\frac{1}{8}x + \frac{25}{8}.$

(b) $f(x) = \sqrt{1 - x^2}$ $\quad \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{-x}{\sqrt{1-x^2}}$ $\quad \frac{\mathrm{d}f}{\mathrm{d}x}\Big|_{x=\frac{\sqrt{2}}{2}} = -1$

tangent line $y = -x + c$. This passes through $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ so

$\frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2} + c \quad \therefore \quad c = \sqrt{2} \quad \therefore \quad y = -x + \sqrt{2}$

normal line $y = x + d$. This passes through $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ so

$\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} + \mathrm{d} \quad \therefore \quad \mathrm{d} = 0 \quad \therefore \quad y = x.$

At (1,0) the tangent line is $x = 1$ and the normal line is $y = 0$.

(c) $f(x) = x^{\frac{1}{2}}$ $\quad \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{1}{2}x^{-\frac{1}{2}}$ $\quad \frac{\mathrm{d}f}{\mathrm{d}x}\Big|_{x=1} = \frac{1}{2}$

tangent line: $y = \frac{1}{2}x + c$. This passes through (1,1) so

$1 = \frac{1}{2} + c \quad \therefore \quad c = \frac{1}{2} \quad \therefore \quad y = \frac{1}{2}x + \frac{1}{2}$

normal line: $y = -2x + \mathrm{d}$. This passes through (1,1) so

$1 = -2 + \mathrm{d} \quad \therefore \quad \mathrm{d} = 3 \quad \therefore \quad y = -2x + 3.$