Maxima and Minima 12.2

Introduction

In this section we analyse curves in the 'local neighbourhood' of a stationary point and, from this analysis, deduce necessary conditions satisfied by local maxima and local minima. Locating the maxima and minima of a function is an important task which arises often in applications of mathematics to problems in engineering and science. It is a task which can often be carried out using only a knowledge of the derivatives of the function concerned. The problem breaks into two parts

- finding the stationary points of the given functions
- distinguishing whether these stationary points are maxima, minima or so-called points of inflection.

Prerequisites

Before starting this Section you should . . .

① be able to take first and second derivatives of simple functions

② be able to find the roots of simple equations

Learning Outcomes

After completing this Section you should be able to . . .

✓ understand the difference between local and global maxima and minima

✓ appreciate how tangent lines change near a maximum or a minimum

✓ locate the position of stationary points

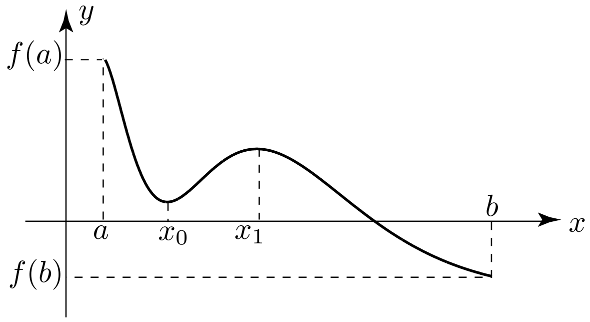
✓ use knowledge of the second derivative to distinguish between maxima and minima

1. Maxima and Minima

Consider the curve

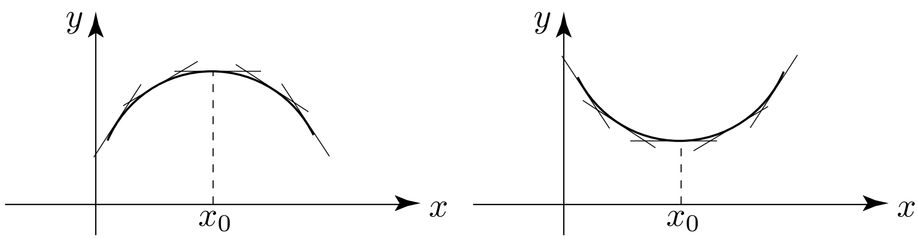
$$y = f(x) \qquad a \le x \le b$$

shown in the following figure



By inspection we see that for this curve there is no $y-$value greater than that at $x = a$ (i.e. $f(a)$) and there is no smaller value than that at $x = b$ (i.e. $f(b)$). However, the points on the curve at x_0 and x_1 merit comment. It is clear that in the **near** neighbourhood of x_0 all the $y-$values are greater than the $y-$value at x_0 and, similarly, in the **close** neighbourhood of x_1 all the $y-$values are less than the $y-$value at x_1.
We say $f(x)$ has a **global maximum** at $x = a$ and a **global minimum** at $x = b$ but also has a **local minimum** at $x = x_0$ and a **local maximum** at $x = x_1$.
Our primary purpose in this section is to see how we might locate the position of the local maxima and the local minima for a smooth function $f(x)$.

A **stationary point** on a curve is one at which the derivative has a zero value. In the following diagram we have sketched a curve with a maximum and a curve with a minimum.



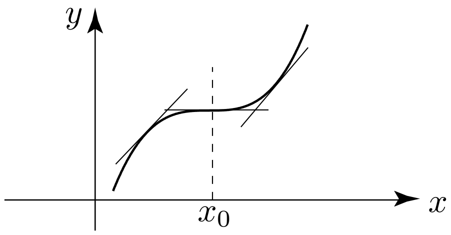
By drawing tangent lines to these curves in the close neighbourhood of the local maximum and the local minimum it is obvious that at these points the tangent line is parallel to the $x-$axis so that

$$\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_0} = 0$$

Key Point

Points on the curve $y = f(x)$ at which $\frac{\mathrm{d}f}{\mathrm{d}x} = 0$ are called **stationary points** of the function

However, be careful! A stationary point is not necessarily a local maximum or minimum of the function but may be an exceptional point called a point of inflection: see diagram.



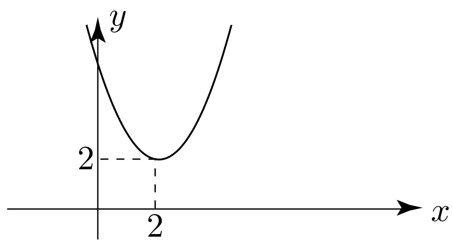
Example Sketch the curve $y = (x-2)^2+2$ and locate the stationary points on the curve.

Solution

Here $f(x) = (x-2)^2 + 2$ so $\frac{df}{dx} = 2(x-2)$.

At a stationary point $\frac{df}{dx} = 0$ so we have $2(x-2) = 0$ so $x = 2$. We conclude that this function has just one stationary point located at $x = 2$ (where $y = 2$).

By sketching the curve $y = f(x)$ it is clear that this stationary point is a local minimum.



Locate the position of any stationary points of $\quad f(x) = x^3 - 1.5x^2 - 6x + 10.$

First find $\frac{df}{dx}$

Your solution

$\frac{df}{dx} =$

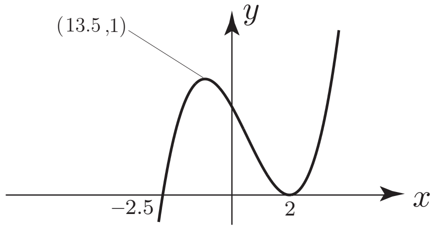
$\frac{df}{dx} = 3x^2 - 3x - 6$

Now locate the stationary points by solving $\frac{df}{dx} = 0$

Your solution

$3x^2 - 3x - 6 = 3(x+1)(x-2) = 0 \quad$ so $x = -1$ or $x = 2$

Finally the $y-$coordinates of the stationary points can be found by substituting these x values into the equation for y. The stationary points are $(0, 2)$ and $(-1, 13.5)$. We have, in the following diagram, sketched the curve which confirms our deductions.

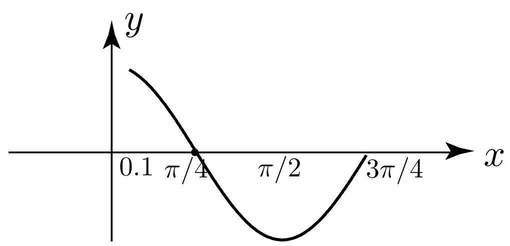


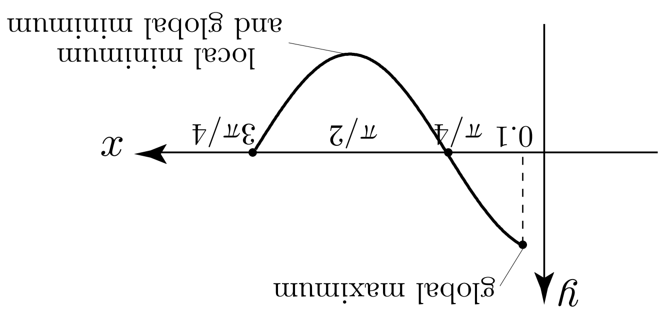
Sketch the curve

$$y = \cos 2x \qquad 0.1 \leq x \leq \frac{3\pi}{4}$$

and locate the position of the global maximum, global minimum and any local maxima and/or minima.

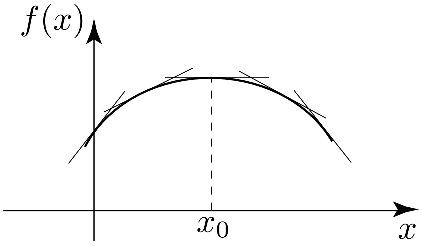
Your solution





2. Distinguishing between local maxima and minima

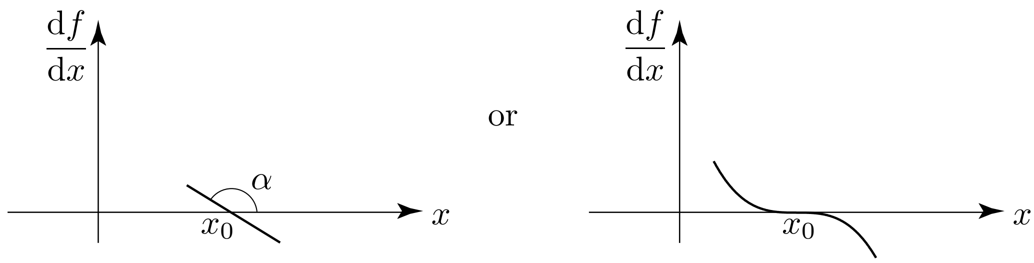
We might ask if it is possible to predict when a stationary point is a local maximum, a local minimum or a point of inflection without the necessity of drawing the curve. To do this we highlight the general characteristics of curves in the neighbourhood of local maxima and minima. For example: at a local maximum (located at x_0 say) the following diagram correctly describes the situation:



to the left of the maximum $\dfrac{df}{dx} > 0$

and to the right of the maximum $\dfrac{df}{dx} < 0$

If we draw a graph of the *derivative* $\frac{df}{dx}$ against x then, near a local maximum, it **must** take one of two basic shapes described in the following diagram:

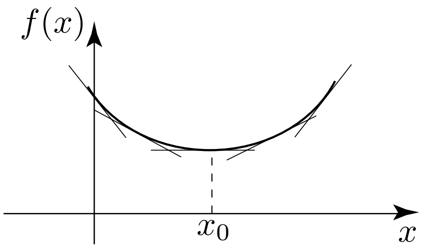


(a) (b)

In case (a) $\quad \dfrac{d}{dx}\left(\dfrac{df}{dx}\right)\Bigg|_{x_0} \equiv \tan\alpha < 0 \quad$ whilst in case (b) $\quad \dfrac{d}{dx}\left(\dfrac{df}{dx}\right)\Bigg|_{x_0} = 0$

We reach the conclusion that at a stationary point which is a maximum the value of the second derivative $\frac{d^2f}{dx^2}$ is either negative or zero.

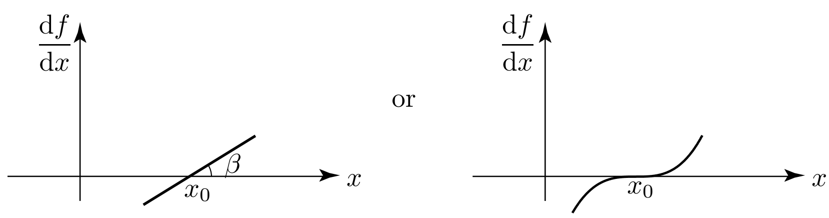
Near a local minimum the above graphs are inverted



to the left of the minimum $\dfrac{df}{dx} < 0$

and to the right of the minimum $\dfrac{df}{dx} > 0$

whilst the graphs of the derivative are either:

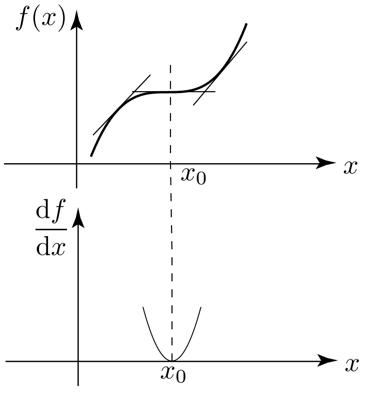


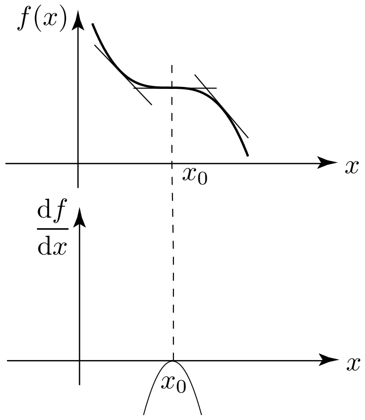
(a) (b)

Here,

$$\text{for case (a)} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\bigg|_{x_0} = \tan\beta > 0 \qquad \text{whilst in (b)} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\bigg|_{x_0} = 0.$$

In this case we conclude that at a stationary point which is a minimum the value of the second derivative $\frac{\mathrm{d}^2f}{\mathrm{d}x^2}$ is either positive or zero. For the remaining possibility for a stationary point, a point of inflection, the graph of $f(x)$ against x and of $\frac{\mathrm{d}f}{\mathrm{d}x}$ against x take one of two forms:





to the left of x_0 $\quad \frac{\mathrm{d}f}{\mathrm{d}x} > 0$ $\qquad$ to the left of x_0 $\quad \frac{\mathrm{d}f}{\mathrm{d}x} < 0$

to the right of x_0 $\quad \frac{\mathrm{d}f}{\mathrm{d}x} > 0$ $\qquad$ to the right of x_0 $\quad \frac{\mathrm{d}f}{\mathrm{d}x} < 0$

$$\text{For either of these cases} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\bigg|_{x_0} = 0$$

The sketches and analysis of the shape of a curve $y = f(x)$ in the near neighbourhood of stationary points allow us to make the following important deduction

Key Point

If x_0 locates a stationary point of the function $f(x)$, so that $\frac{\mathrm{d}f}{\mathrm{d}x}\Big|_{x_0} = 0$, then the stationary point is a local minimum if

$$\frac{\mathrm{d}^2f}{\mathrm{d}x^2}\bigg|_{x_0} > 0$$

and is a local maximum if

$$\frac{\mathrm{d}^2f}{\mathrm{d}x^2}\bigg|_{x_0} < 0$$

and is inconclusive if $\frac{\mathrm{d}^2f}{\mathrm{d}x^2}\Big|_{x_0} = 0$

Example Find the stationary points of the function

$$f(x) = x^3 - 6x$$

Are these stationary points local maxima or local minima?

Solution

$\frac{df}{dx} = 3x^2 - 6$

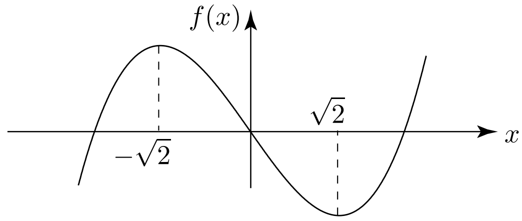
At a stationary point $\frac{df}{dx} = 0$ so $3x^2 - 6 = 0$ implying $x = \pm\sqrt{2}$.

Thus $f(x)$ has stationary points at $x = \sqrt{2}$ and $x = -\sqrt{2}$. To see if these are maxima or minima we examine the value of the second derivative of $f(x)$ at the stationary points.

Now $\frac{d^2f}{dx^2} = 6x$ so $\left.\frac{d^2f}{dx^2}\right|_{x=\sqrt{2}} = 6\sqrt{2} > 0$. Hence $x = \sqrt{2}$ locates a local minimum.

Similarly $\left.\frac{d^2f}{dx^2}\right|_{x=-\sqrt{2}} = -6\sqrt{2} < 0$. Hence $x = -\sqrt{2}$ locates a local maximum.

A simple sketch of the curve confirms this analysis.



For the function $f(x) = \cos 2x, \quad 0.1 \le x \le 6$, find the positions of any local minima and/or maxima and distinguish between each kind.

Your solution

$\frac{df}{dx} =$

$\therefore$ stationary points are located at:

$\frac{df}{dx} = -2\sin 2x$.

Hence stationary points are at values of x for which $\sin 2x = 0$ i.e. at $2x = \pi$ or $2x = 2\pi$ or $2x = 3\pi$ (make sure x is within the range $0.1 \le x \le 6$

$\therefore$ possible stationary points at $x = \frac{\pi}{2}$, $x = \pi$, $x = \frac{3\pi}{2}$

Now calculate the second derivative

Your solution

$\frac{d^2f}{dx^2} =$

$\dfrac{d^2f}{dx^2} = -4\cos 2x$

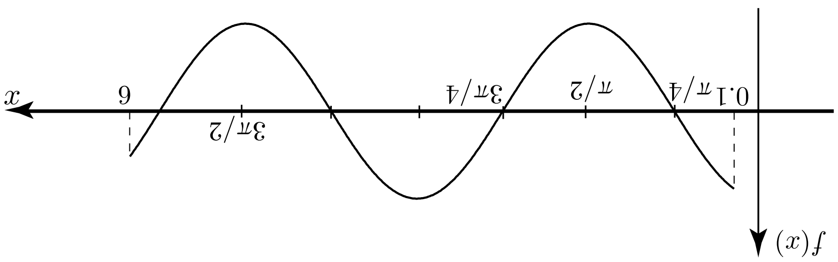
Finally: evaluate the second derivative at each of the stationary points and draw the appropriate conclusions.

Your solution

$$\left.\frac{d^2f}{dx^2}\right|_{x=\frac{\pi}{2}} =$$

$$\left.\frac{d^2f}{dx^2}\right|_{x=\pi} =$$

$$\left.\frac{d^2f}{dx^2}\right|_{x=\frac{3\pi}{2}} =$$



$$\left.\frac{d^2f}{dx^2}\right|_{x=\frac{\pi}{2}} = -4\cos\pi = 4 > 0 \quad \therefore \quad x = \tfrac{\pi}{2} \text{ locates a local minimum.}$$

$$\left.\frac{d^2f}{dx^2}\right|_{x=\pi} = -4\cos 2\pi = -4 < 0 \quad \therefore \quad x = \pi \text{ locates a local maximum.}$$

$$\left.\frac{d^2f}{dx^2}\right|_{x=\frac{3\pi}{2}} = -4\cos 3\pi = 4 > 0 \quad \therefore \quad x = \tfrac{3\pi}{2} \text{ locates a local minimum.}$$

Determine the local maxima and/or minima of the function

$$y = x^4 - \frac{1}{3}x^3$$

First obtain the positions of the stationary points.

Your solution

$f(x) = x^4 - \frac{1}{3}x^3 \qquad \dfrac{df}{dx} =$

Thus $\dfrac{df}{dx} = 0$ when:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 4x^3 - x^2 = x^2(4x-1)$$

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 0 \text{ when } x = 0 \text{ or when } x = \frac{1}{4}$$

Now obtain the value of the second derivatives at the stationary points

Your solution

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} =$$

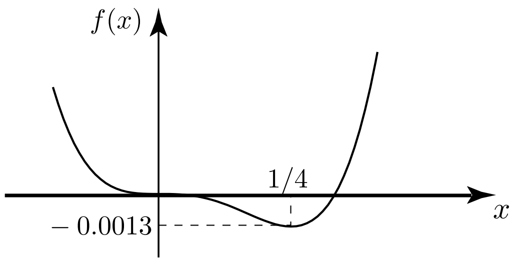
$$\therefore \quad \left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_{x=0} \qquad\qquad \left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_{x=\frac{1}{4}} =$$

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = 12x^2 - 2x$$

$$\left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_{x=0} = 0, \qquad \left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_{x=\frac{1}{4}} = \frac{12}{16} - \frac{1}{2} = \frac{1}{4} > 0$$

Hence $x = \frac{1}{4}$ locates a local minimum.

However, using this analysis we cannot conclude that the stationary point at $x = 0$ is a local maximum, minimum or a point of inflection. However just to the left of $x = 0$ the value of $\frac{\mathrm{d}f}{\mathrm{d}x}$ (which equals $x^2(4x-1)$) is $-$ve whilst just to the right of $x = 0$ the value of $\frac{\mathrm{d}f}{\mathrm{d}x}$ is $-$ve. Hence the stationary point at $x = 0$ is a point of inflection. This is confirmed by sketching the curve.



Exercises

Locate the stationary points of the following functions and distinguish between maxima and minima.

(a) $f(x) = x - \ln|x|$. Remember $\frac{\mathrm{d}}{\mathrm{d}x}(\ln|x|) = \frac{1}{x}$.

(b) $f(x) = x^3$

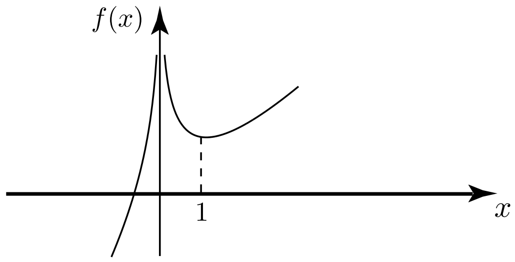
(c) $f(x) = \frac{(x-1)}{(x+1)(x-2)} \qquad -1 < x < 2$

Answers

(a) $\frac{\mathrm{d}f}{\mathrm{d}x} = 1 - \frac{1}{x} = 0$ when $x = 1$

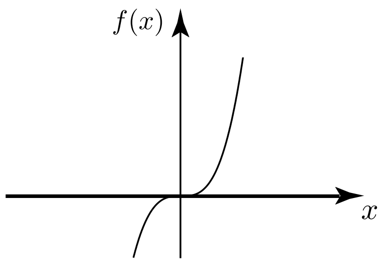
$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = \frac{1}{x^2} \qquad \left.\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_{x=1} = 1 > 0$

$\therefore \quad x = 1, y = 1$ locates a local minimum



(b) $\frac{\mathrm{d}f}{\mathrm{d}x} = 3x^2 = 0$ when $x = 0 \qquad \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = 6x = 0$ when $x = 0$

However, $\frac{\mathrm{d}f}{\mathrm{d}x} > 0$ on either side of $x = 0$ so (0,0) is a point of inflection.



(c) $\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{(x+1)(x-2)-(x-1)[2x-1]}{(x+1)(x-2)}$

This is zero when $(x+1)(x-2) - (x-1)(2x-1) = 0$

i.e. $x^2 - 2x + 3 = 0$

However, this equation has no real roots ($b^2 < 4ac$) and so $f(x)$ has no stationary points. The graph of this function confirms this.