The Newton-Raphson Method

12.3

Introduction

This section is concerned with the problem of "root location"; i.e. finding those values of x which satisfy an equation of the form $f(x) = 0$ for a given function $f(x)$. An initial estimate of the root is found by drawing a graph of the function in the neighbourhood of the root. This estimate is then improved by using a technique known as the Newton-Raphson method. The method is based upon a knowledge of the tangent to the curve near the root. It is an "iterative" method in that it can be used repeatedly to continually improve the accuracy of the root.

Prerequisites

Before starting this Section you should ...

① be able to differentiate simple functions

② be able to sketch graphs

Learning Outcomes

After completing this Section you should be able to ...

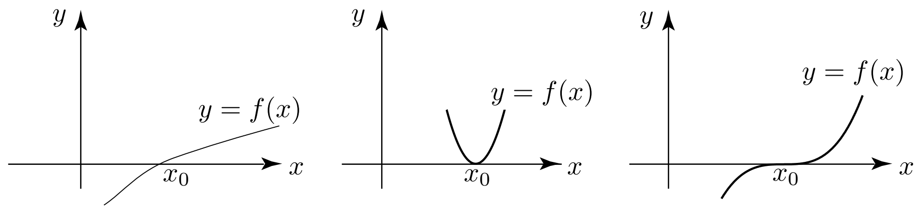
✓ distinguish between simple and multiple roots

✓ estimate the root the root of an equation by drawing a graph

✓ employ the Newton-Raphson method to improve the accuracy of a root

1. The Newton-Raphson Method

We first remind the reader of some basic notation: If $f(x)$ is a given function the value of x for which $f(x) = 0$ is called a **root** or **zero** of the function. We also distinguish between various types of roots: Simple and multiple roots. The following diagram illustrates some common examples.



simple root double root triple root

More precisely; a root x_0 is said to be:

a **simple root** if $\quad f(x_0) = 0 \quad$ and $\quad \left.\dfrac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_0} \neq 0.$

a **double root** if $\quad f(x_0) = 0, \quad \left.\dfrac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_0} = 0 \quad$ and $\quad \left.\dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right|_{x_0} \neq 0.$

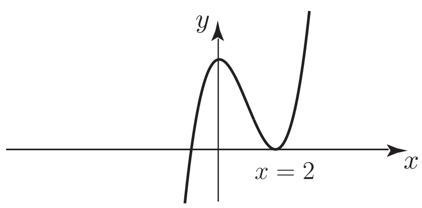
and so on.

In this section we shall concentrate on the location of simple roots of a given function $f(x)$

Plot the functions (i) $f(x) = x^3 - 3x^2 + 4$, (ii) $f(x) = 1 + \sin x$ and classify the roots into simple or multiple.

Your solution

Sketch $f(x) = x^3 - 3x^2 + 4$

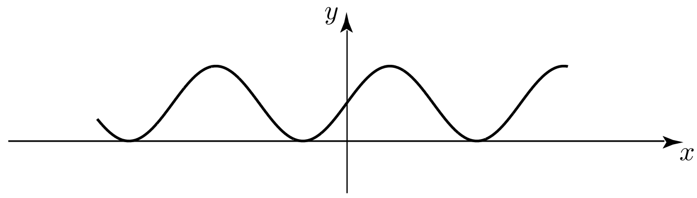


The negative root is and the positive root is

The negative root is simple and the positive root is double.

Your solution

Sketch $f(x) = 1 + \sin x$



Each root is a root

Each root is a double root

2. Finding Roots of the equation $f(x) = 0$

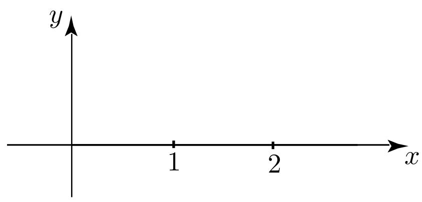
A first investigation into the roots of $f(x)$ would be graphical. Such an analysis will supply information as to the approximate location of the roots.

Sketch the function

$$f(x) = x - 2 + \ln x \qquad x > 0$$

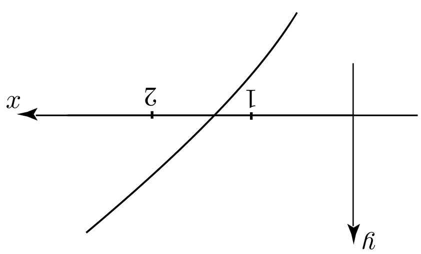
and estimate the value of the root.

Your solution



An estimate of the root is

A simple root is located near 1.5



One method of obtaining a better approximation is to halve the interval $1 \leq x \leq 2$ into $1 \leq x \leq 1.5$ and $1.5 \leq x \leq 2$ and test the **sign** of the function at the end-points of these new regions. We find

x	$f(x)$
1	< 0
1.5	< 0
2	> 0

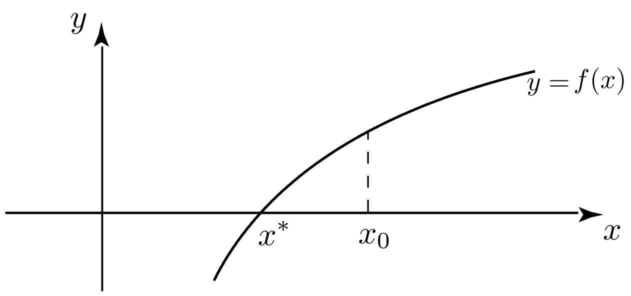
so the root we seek must lie between $x = 1.5$ and $x = 2$ because the sign of $f(x)$ changes between these values.

We can repeat this procedure and divide the interval (1.5,2) into the two new intervals (1.5,1.75) and (1.75,2) and test again. This time we find

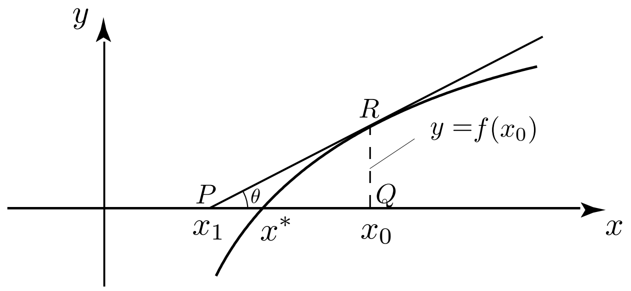
x	$f(x)$
1.5	< 0
1.75	> 0
2.0	> 0

so the root lies in the interval (1.5,1.75). It is obvious that proceeding in this way will give a smaller and smaller interval in which the root must lie. But can we do better than this rather laborious bisection procedure? In fact there are many ways to improve this numerical search for the root. In this section we examine one of the best methods: the **Newton-Raphson** method. To obtain the method we examine the general characteristics of a curve in the neighbourhood of a simple root.

Consider the following diagram showing a function $f(x)$ with a simple root at $x = x^*$ whose value is required. Initial analysis has indicated that the root is approximately located at $x = x_0$. The aim of any numerical procedure is to provide a better estimate to the location of the root.



The basic premise of the Newton-Raphson method is the assumption that the curve in the close neighbourhood of the simple root at x^* is approximately a straight line. Hence if we draw the tangent to the curve at x_0, this tangent will intersect the $x-$axis at a point closer to x^* than is x_0: see the following diagram.



From the geometry of this diagram we see that

$$x_1 = x_0 - PQ$$

But from the right-angled triangle PQR we have

$$\frac{RQ}{PQ} = \tan\theta = f'(x_0)$$

and so

$$PQ = \frac{RQ}{[f'x_0]} = \frac{f(x_0)}{f'(x_0)}$$

$$\therefore \qquad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

If $f(x)$ has a simple root near x_0 then a closer estimate to the root is x_1 where

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

This formula can be used time and time again giving rise to the key point:

Key Point

If $f(x)$ has a simple root near x_n then a closer estimate to the root is x_{n+1} where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This is the Newton-Raphson iterative formula. The iteration is begun with an initial estimate of the root, x_0, and continued to find $x_1, x_2, \ldots$ until a suitably accurate estimate of the position of the root is obtained.

Example $f(x) = x - 2 + \ln x$ has a root near $x = 1.5$. Use the Newton-Raphson formula to obtain a better estimate.

Solution

Here $x_0 = 1.5$, $\quad f(1.5) = -0.5 + \ln(1.5) = -0.0945$

$f'(x) = 1 + \frac{1}{x} \qquad \therefore \qquad f'(1.5) = 1 + \frac{1}{1.5} = \frac{5}{3}$

Hence using the formula:

$$x_1 = 1.5 - \frac{(-0.0945)}{(1.6667)} = 1.5567$$

The Newton-Raphson formula can be used again: this time beginning with 1.5567 as our initial estimate.

This time use:

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= 1.5567 - \frac{f(1.5567)}{f'(1.5567)} \\ &= 1.5567 - \frac{\{1.5567 - 2 + \ln(1.5567)\}}{\{1 + \frac{1}{1.5567}\}} \\ &= 1.5567 - \frac{\{-0.0007\}}{\{1.6424\}} = 1.5571 \end{aligned}$$

This is in fact the correct value of the root to 4 d.p.

The function $f(x) = x - \tan x$ has a simple root near $x = 4.5$. Use one iteration of the Newton-Raphson method to find a more accurate value for the root.

First find $\frac{df}{dx}$

Your solution

$$\frac{df}{dx} =$$

$\frac{df}{dx} = 1 - \sec^2 x = -\tan^2 x$

Now use the formula $x_1 = x_0 - f(x_0)/f'(x_0)$ with $x_0 = 4.5$ to obtain x_1.

Your solution

$f(4.5) = 4.5 - \tan 4.5 =$

$f'(4.5) = 1 - \sec^2 4.5 = -\tan^2 4.5 =$

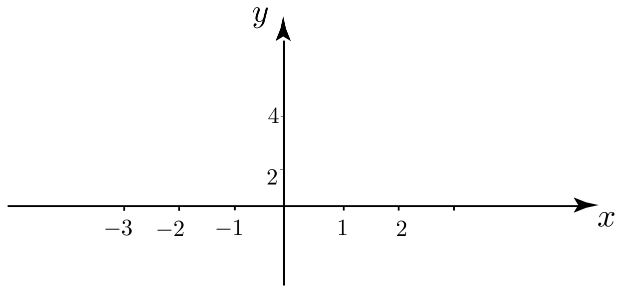
$x_1 = 4.5 - \frac{f(4.5)}{f'(4.5)} =$

$f(4.5) = -0.1373, \quad f'(4.5) = -21.5048$

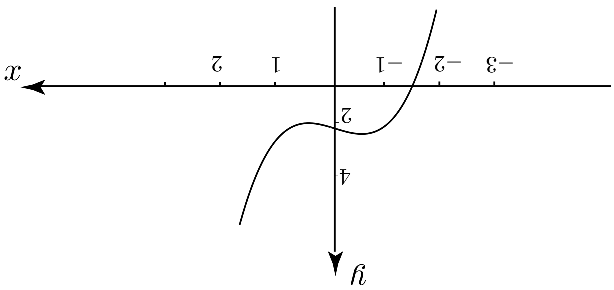
$\therefore \quad x_1 = 4.5 - \frac{0.1373}{21.5048} = 4.4936$

Sketch the function $f(x) = x^3 - x + 3$ and confirm that there is a simple root near $x = -2$. Using this as an initial estimate obtain the value of the simple root accurate to 2 d.p.

Your solution



Clearly a simple root near $x = -2$



Now use Newton-Raphson to improve your estimate

Your solution

$f(x) =$ $\qquad$ $f'(x) =$ $\qquad$ $x_0 =$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} =$$

$f(x) = x^3 - x + 3, \quad f'(x) = 3x^2 - 1 \quad x_0 = -2$

$\therefore \quad x_1 = -2 - \frac{\{-8+2+3\}}{11} = -2 + \frac{3}{11} = -1.727$

Now repeat this process

Your solution

$x_2 = x_1 - f(x_1)/f'(x_1) =$

$$\begin{aligned} x_2 &= -1.727 - \{-(1.727)^3 + 1.727 + 3\}/[3(1.727)^2 - 1] \\ &= -1.727 + \{(0.424)/(7.948) = -1.674 \end{aligned}$$

Repeating again

Your solution

$x_3 = x_2 - f(x_2)/f'(x_2) =$

$$\begin{aligned} x_3 &= -1.674 - \{-(1.674)^3 + 1.674 + 3\}/\{3(1.674)^2 - 1\} \\ &= -1.674 + \{0.017\}/\{7.407\} = -1.672 \end{aligned}$$

We conclude the value of the simple root is -1.67 correct to 2 d.p.

Exercises

1. By sketching the function $f(x) = x - 1 - \sin x$ show that there is a simple root near $x = 2$. Use two steps of the Newton-Raphson method to obtain a better estimate of the root.
2. Use the Newton-Raphson procedure to find $\sqrt{2}$ correct to 3 d.p. (Hint: Solve the equation $x^2 - 2 = 0$ with initial estimate: 1.4 (obtained by drawing the graph $y = x^2 - 2$)).
3. Obtain an estimation accurate to 2 d.p. of the point of intersection of the curves $y = x - 1$ and $y = \cos x$.

Answers

1. $[x_0 = 2, \quad x_1 = 1.936, \quad x_2 = 1.935]$
2. $[x_0 = 1.4, \quad x_1 = 1.4143, \quad x_2 = 1.4142]$
3. The curves intersect when $x - 1 - \cos x = 0$. Solve this using the Newton-Raphson method with initial estimate (say) $x_0 = 1.2$.

   The point of intersection is $(1.28342, \ 0.283437)$