Curvature

12.4

Introduction

Curvature is a measure of how sharply a curve is turning as it is traversed. At a particular point along the curve a tangent line can be drawn; this line making an angle ψ with the positive $x-$axis. Curvature is then defined as the magnitude of the rate of change of ψ with respect to the measure of length on the curve - the arc length s. That is

$$\text{Curvature } = \left|\frac{\mathrm{d}\psi}{\mathrm{d}s}\right|$$

In this section we examine the concept of curvature and, from its definition, obtain more useful expressions when the equation of the curve is expressed either in Cartesian form $y = f(x)$ or in parametric form $x = x(t) \quad y = y(t)$. We show that a circle has a constant value for the curvature, which is to be expected, as the tangent line to a circle turns equally quickly irrespective of the position on the circle. For every other curve, other than a circle, the curvature will depend upon position, changing its value as the curve twists and turns.

Prerequisites

Before starting this Section you should . . .

① understand the geometrical interpretation of the derivative

② be able to differentiate standard functions

③ be able to use the parametric description of a curve

Learning Outcomes

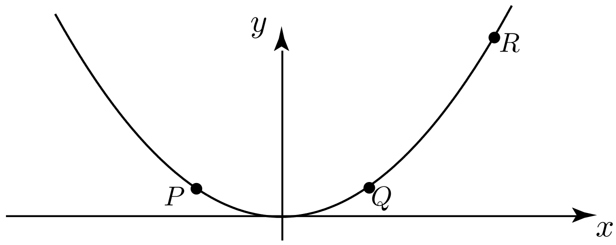
After completing this Section you should be able to . . .

✓ understand the concept of curvature

✓ be able to calculate curvature if the curve is defined in Cartesian form or in parametric form

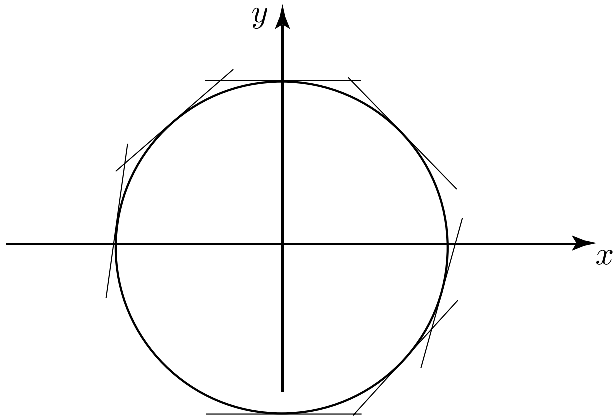
1. Curvature

Curvature is a measure of how quickly a tangent line turns on a curve. For example, consider a simple parabola, with equation $y = x^2$. Its graph is shown in the following diagram.

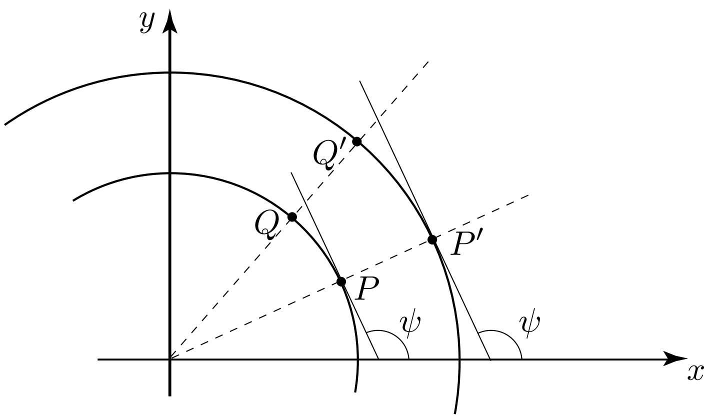


It is obvious, geometrically, that the tangent lines to this curve turn 'more quickly' between P and Q than along the curve from Q to R. It is the purpose of this section to give, a quantitative measure of this rate of 'turning'.

If we change from a parabola to a circle, (centred on the origin, of radius 1) we can again consider how quickly the tangent lines turn as we move along the curve. It is immediately clear that the tangent lines to a circle turn equally quickly no matter where on the circle you choose to consider.



However, if we consider two circle with the same centre but different radii:



It is again obvious that the smaller circle 'bends' more tightly than the larger circle and we say it has a larger curvature. Athletes who run the 200 metres find it easier to run in the outside lanes (where the curve turns less sharply) than in the inside lanes.

On the two circle diagram (above) we have drawn tangent lines at P and P'; both lines make an angle ψ with the positive $x-$axis. We need to measure how quickly the angle ψ changes as we move along the curve. As we, move, from P to Q (inner circle) or from P' to Q' (outer circle) the angle ψ changes by the same amount. However, the distance traversed on the inner circle is less than the distance traversed on the outer circle. This suggests that a measure of curvature is:

curvature is the magnitude of the rate of change of ψ with respect to the distance moved along the curve

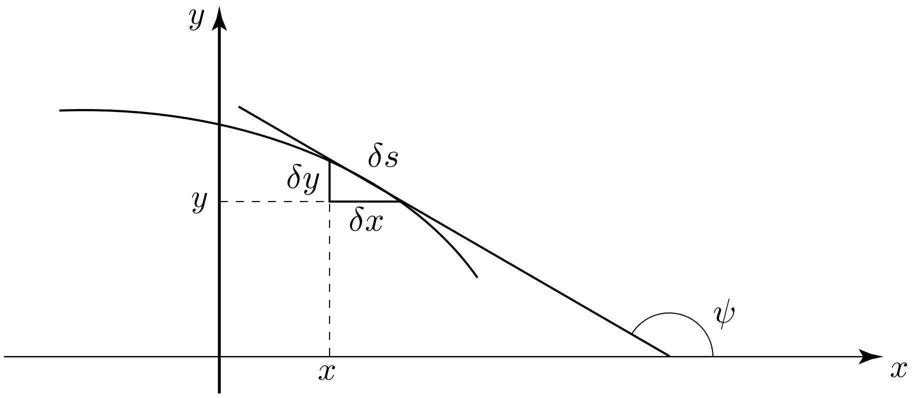
We shall denote the curvature by the greek symbol κ (kappa).
So

$$\kappa = \left|\frac{\mathrm{d}\psi}{\mathrm{d}s}\right|$$

where s is the measure of arc-length along a curve.
This rather odd-looking derivative needs converting to a more familiar form if the equation of the curve is given in the form $y = f(x)$. As a preliminary we note that

$$\frac{\mathrm{d}\psi}{\mathrm{d}s} = \frac{\mathrm{d}\psi}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\mathrm{d}\psi}{\mathrm{d}x}\Big/\left(\frac{\mathrm{d}s}{\mathrm{d}x}\right)$$

We now obtain expressions for the derivatives $\frac{\mathrm{d}\psi}{\mathrm{d}x}$ and $\frac{\mathrm{d}s}{\mathrm{d}x}$ in terms of the derivatives of $f(x)$.
Consider the following diagram.



Small increments in the $x-$ and $y-$directions have been denoted by δx and δy, respectively. The hypotenuse on this 'small' triangle is δs which is the change in arc-length along the curve. From Pythagoras' theorem:

$$\delta s^2 = \delta x^2 + \delta y^2$$

so

$$\left(\frac{\delta s}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2 \qquad \text{so that} \qquad \frac{\delta s}{\delta x} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}$$

In the limit as the increments get smaller and smaller, we write this relation in derivative form:

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$$

However as $y = f(x)$ is the equation of the curve we obtain

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1 + \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2} = (1 + [f'(x)]^2)^{1/2}$$

We also know the relation between the angle ψ and the derivative $\frac{\mathrm{d}f}{\mathrm{d}x}$:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \tan\psi$$

so differentiating again:

$$\begin{aligned}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} &= \sec^2\psi \frac{\mathrm{d}\psi}{\mathrm{d}x} = (1 + \tan^2\psi)\frac{\mathrm{d}\psi}{\mathrm{d}x} \\ &= (1 + [f'(x)]^2)\frac{\mathrm{d}\psi}{\mathrm{d}x}\end{aligned}$$

Inverting this relation:

$$\frac{\mathrm{d}\psi}{\mathrm{d}x} = \frac{f''(x)}{(1 + [f'(x)]^2)}$$

and so, finally, the curvature is given by

$$\kappa = \left|\frac{\mathrm{d}\psi}{\mathrm{d}s}\right| = \left|\frac{\mathrm{d}\psi}{\mathrm{d}x} \Big/ \left(\frac{\mathrm{d}s}{\mathrm{d}x}\right)\right| = \left|\frac{f''(x)}{(1 + [f'(x)]^2)^{3/2}}\right|$$

Key Point

At each point on a curve, with equation $y = f(x)$, the tangent line turns at a certain rate. A measure of this rate of turning is the **curvature** κ

$$\kappa = \left|\frac{f''(x)}{(1 + [f'(x)]^2)^{3/2}}\right|$$

Obtain the curvature of the parabola $y = x^2$. First calculate the derivatives of $f(x)$

Your solution

$f(x) =$ $\qquad$ $\frac{\mathrm{d}f}{\mathrm{d}x} =$ $\qquad$ $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$

$f(x) = x^2$ $\qquad$ $\frac{\mathrm{d}f}{\mathrm{d}x} = 2x$ $\qquad$ $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = 2$

Now find an expression for the curvature

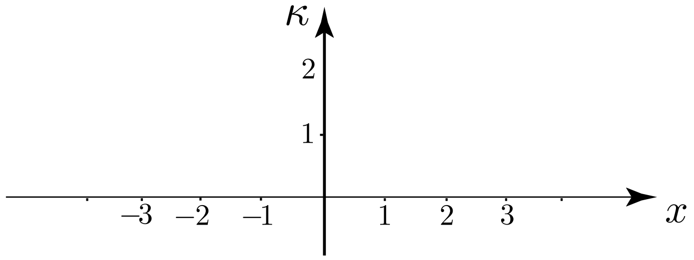
Your solution

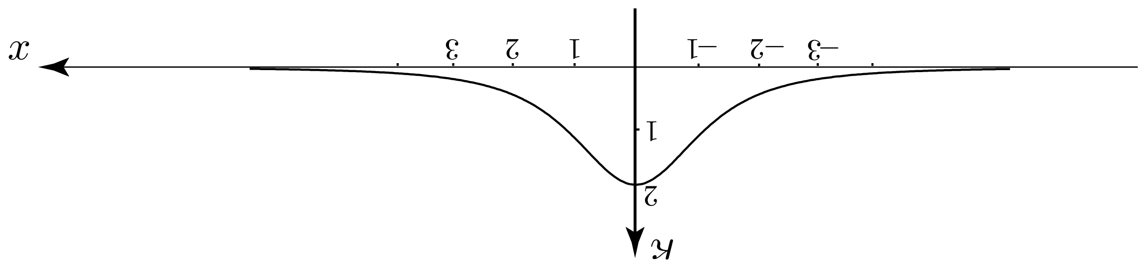
$\kappa =$

$$\kappa = \left| \frac{f''(x)}{[1 + (f'(x))^2]^{3/2}} \right| = \frac{2}{[1 + 4x^2]^{3/2}}$$

Finally, plot the curvature κ as a function of x

Your solution





This picture confirms what we have already argued: close to $x = 0$ the parabola turns sharply. (near $x = 0$ the curvature κ is, relatively, large.) Further away from $x = 0$ the curve is more 'gentle' (in these regions κ is small).

In general, the curvature κ, is a function of position. However, from what we have said earlier, we expect the curvature to be a constant for a given circle but to increase as the radius decreases. This can now be checked directly.

Example Find the curvature of the circle $y = (a^2 - x^2)^{1/2}$ (this is the equation of the upper half of a circle centred at the origin of radius a).

Solution

Here $f(x) = (a^2 - x^2)^{\frac{1}{2}}$

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{-x}{(a^2 - x^2)^{\frac{1}{2}}} \qquad \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = \frac{-a^2}{(a^2 - x^2)^{\frac{3}{2}}}$$

$$\therefore \qquad 1 + [f'(x)]^2 = 1 + \frac{x^2}{a^2 - x^2} = \frac{r^2}{a^2 - x^2}$$

$$\therefore \qquad \kappa = \left| \frac{\dfrac{-a^2}{(a^2 - x^2)^{3/2}}}{\left[\dfrac{a^2}{a^2 - x^2}\right]^{3/2}} \right| = \frac{1}{a}$$

For a circle, the curvature is constant.

The value of κ (at any particular point on the curve, i.e. at a particular value of x) indicates how sharply the curve is turning. What this result states is that, for a circle, the curvature is inversely related to the radius. The bigger the radius, the smaller the curvature; precisely what, as we have argued above, we should expect.

2. Curvature for parametrically defined curves

An expression for the curvature is also available if the curve is described parametrically:

$$x = g(t) \qquad y = h(t) \qquad t_0 \le t \le t_1$$

We remember the basic formulae connecting derivatives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$$

where, as usual $\dot{x} \equiv \frac{\mathrm{d}x}{\mathrm{d}t}$, $\quad \ddot{x} \equiv \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$ etc.

Then

$$\kappa = \left| \frac{f''(x)}{\{1 + [f'(x)]^2\}^{3/2}} \right| = \left| \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3 \left[1 + \left(\frac{\dot{y}}{\dot{x}}\right)^2\right]^{3/2}} \right|$$

$$= \left| \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{[\dot{x}^2 + \dot{y}^2]^{3/2}} \right|$$

Key Point

The formula for the curvature in parametric form is

$$\kappa = \left| \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{[\dot{x}^2 + \dot{y}^2]^{3/2}} \right|$$

An ellipse is described parametrically by the equations

$$x = 2\cos t \qquad y = \sin t \qquad 0 \leq t \leq 2\pi$$

Obtain an expression for the curvature κ and find where the curvature is a maximum or a minimum.

First find $\dot{x}$, $\dot{y}$, $\ddot{x}$, $\ddot{y}$

Your solution

$\dot{x} =$ $\qquad$ $\dot{y} =$ $\qquad$ $\ddot{x} =$ $\qquad$ $\ddot{y} =$

$\dot{x} = -2\sin t \qquad \dot{y} = \cos t \qquad \ddot{x} = -2\cos t \qquad \ddot{y} = -\sin t$

Now find κ

Your solution

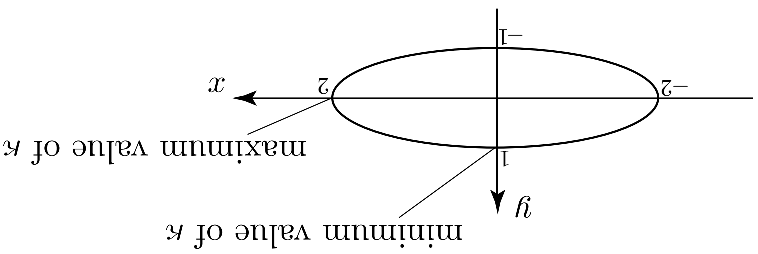
$\kappa =$

$$\kappa = \left| \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{[\dot{x}^2 + \dot{y}^2]^{3/2}} \right| = \left| \frac{2\sin^2 t + 2\cos^2 t}{[4\sin^2 t + \cos^2 t]^{3/2}} \right| = \frac{2}{[1 + 3\sin^2 t]^{3/2}}$$

Find maximum and minimum values of κ **by inspection**

Your solution

$\max \kappa =$ $\qquad$ $\min \kappa =$



denominator is max when $t = \pi/2$. This gives minimum value of $\kappa = 1/4$, denominator is min when $t = 0$. This gives maximum value of $\kappa = 2$