Differentiation of vectors

12.5

Introduction

The area known as vector calculus is used to model mathematically a vast range of engineering phenomena including electrostatics, electromagnetic fields, air flow around aircraft and heat flow in nuclear reactors. In this section we introduce briefly the differential calculus of vectors.

Prerequisites

Before starting this Section you should . . .

① have a knowledge of vectors, in cartesian form

② be able to calculate the scalar and vector products of two vectors

③ be able to differentiate and integrate scalar functions.

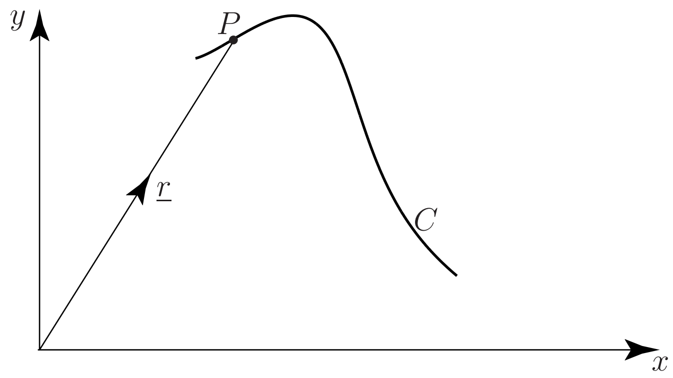
Learning Outcomes

After completing this Section you should be able to . . .

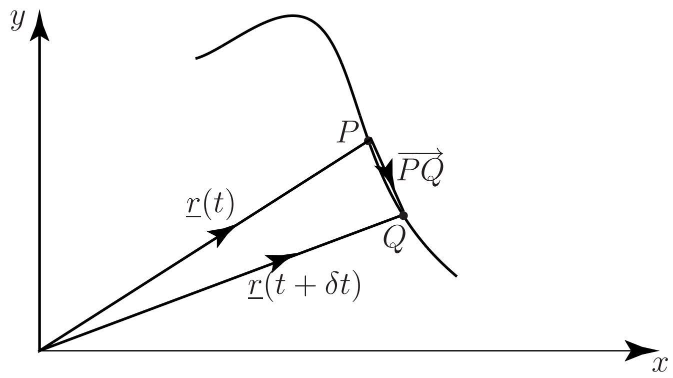
✓ differentiate and integrate vectors

1. Differentiation of Vectors

Consider the following figure.



If $\underline{r}$ represents the position vector of an object which is moving along a curve C, then the position vector will be dependent upon the time, t. We write $\underline{r} = \underline{r}(t)$ to show the dependence upon time. Suppose that the object is at the point P with position vector $\underline{r}$ at time t and at the point Q with position vector $\underline{r}(t + \delta t)$ at the later time $t + \delta t$ as shown in the next figure.



Then $\overrightarrow{PQ}$ represents the displacement vector of the object during the interval of time δt. The length of the displacement vector represents the distance travelled while its direction gives the direction of motion. The average velocity during the time from t to $t + \delta t$ is defined as the displacement vector divided by the time interval δt, that is,

$$\text{average velocity } = \frac{\overrightarrow{PQ}}{\delta t} = \frac{\underline{r}(t+\delta t) - \underline{r}(t)}{\delta t}$$

If we now take the limit as the interval of time δt tends to zero then the expression on the right hand side is the *derivative* of $\underline{r}$ with respect to t. Not surprisingly we refer to this derivative as *the instantaneous velocity*, $\underline{v}$. By its very construction we see that the velocity vector is always tangential to the curve as the object moves along it. We have:

$$\underline{v} = \lim_{\delta t \to 0} \frac{\underline{r}(t+\delta t) - \underline{r}(t)}{\delta t} = \frac{\mathrm{d}\underline{r}}{\mathrm{d}t}$$

Now, since the x and y coordinates of the object depend upon the time, we can write the position vector $\underline{r}$ in cartesian coordinates as:

$$\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j}$$

Therefore,

$$\underline{r}(t+\delta t) = x(t+\delta t)\underline{i} + y(t+\delta t)\underline{j}$$

so that,

$$\begin{aligned}\underline{v}(t) &= \lim_{\delta t\to 0} \frac{x(t+\delta t)\underline{i} + y(t+\delta t)\underline{j} - x(t)\underline{i} - y(t)\underline{j}}{\delta t} \\ &= \lim_{\delta t\to 0} \left\{ \frac{x(t+\delta t) - x(t)}{\delta t}\underline{i} + \frac{y(t+\delta t) - y(t)}{\delta t}\underline{j} \right\} \\ &= \frac{\mathrm{d}x}{\mathrm{d}t}\underline{i} + \frac{\mathrm{d}y}{\mathrm{d}t}\underline{j}\end{aligned}$$

This is often abbreviated to $\underline{v} = \dot{\underline{r}} = \dot{x}\underline{i} + \dot{y}\underline{j}$ using notation for derivatives with respect to time.
So the velocity vector is the derivative of the position vector with respect to time.
This result generalizes in an obvious way to three dimensions. If,

$$\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$$

then the velocity vector is

$$\underline{v} = \dot{\underline{r}}(t) = \dot{x}(t)\underline{i} + \dot{y}(t)\underline{j} + \dot{z}(t)\underline{k}$$

The magnitude of the velocity vector gives the speed of the object.
We can define the acceleration vector in a similar way, as the rate of change (i.e. the derivative) of the velocity with respect to the time:

$$\underline{a} = \frac{\mathrm{d}\underline{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\underline{r}}{\mathrm{d}t^2} = \ddot{\underline{r}} = \ddot{x}\underline{i} + \ddot{y}\underline{j} + \ddot{z}\underline{k}$$

Example If $\underline{w} = 3t^2\underline{i} + \cos 2t\underline{j}$, find

(a) $\dfrac{\mathrm{d}\underline{w}}{\mathrm{d}t}$ (b) $\left|\dfrac{\mathrm{d}\underline{w}}{\mathrm{d}t}\right|$ (c) $\dfrac{\mathrm{d}^2\underline{w}}{\mathrm{d}t^2}$

Solution

1. (a) If $\underline{w} = 3t^2\underline{i} + \cos 2t\underline{j}$, then differentiation with respect to t yields: $\dfrac{\mathrm{d}\underline{w}}{\mathrm{d}t} = 6t\underline{i} - 2\sin 2t\underline{j}$

   (b) $\left|\dfrac{\mathrm{d}\underline{w}}{\mathrm{d}t}\right| = \sqrt{(6t)^2 + (-2\sin 2t)^2} = \sqrt{36t^2 + 4\sin^2 2t}$

   (c) $\dfrac{\mathrm{d}^2\underline{w}}{\mathrm{d}t^2} = 6\underline{i} - 4\cos 2t\underline{j}$

It is possible to differentiate more complicated expressions involving vectors provided certain rules are adhered to. If $\underline{w}$ and $\underline{z}$ are vectors and c is a scalar, all these being functions of time t, then:

Key Point

$$\frac{\mathrm{d}}{\mathrm{d}t}(\underline{w}+\underline{z}) = \frac{\mathrm{d}\underline{w}}{\mathrm{d}t} + \frac{\mathrm{d}\underline{z}}{\mathrm{d}t} \qquad \frac{\mathrm{d}}{\mathrm{d}t}(c\underline{w}) = c\frac{\mathrm{d}\underline{w}}{\mathrm{d}t} + \frac{\mathrm{d}c}{\mathrm{d}t}\underline{w}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\underline{w}\cdot\underline{z}) = \underline{w}\cdot\frac{\mathrm{d}\underline{z}}{\mathrm{d}t} + \frac{\mathrm{d}\underline{w}}{\mathrm{d}t}\cdot\underline{z} \qquad \frac{\mathrm{d}}{\mathrm{d}t}(\underline{w}\times\underline{z}) = \underline{w}\times\frac{\mathrm{d}\underline{z}}{\mathrm{d}t} + \frac{\mathrm{d}\underline{w}}{\mathrm{d}t}\times\underline{z}$$

Example If $\underline{w} = 3t\underline{i} - t^2\underline{j}$ and $\underline{z} = 2t^2\underline{i} + 3\underline{j}$, verify the results

(a) $\frac{\mathrm{d}}{\mathrm{d}t}(\underline{w}\cdot\underline{z}) = \underline{w}\cdot\frac{\mathrm{d}\underline{z}}{\mathrm{d}t} + \frac{\mathrm{d}\underline{w}}{\mathrm{d}t}\cdot\underline{z}$ (b) $\frac{\mathrm{d}}{\mathrm{d}t}(\underline{w}\times\underline{z}) = \underline{w}\times\frac{\mathrm{d}\underline{z}}{\mathrm{d}t} + \frac{\mathrm{d}\underline{w}}{\mathrm{d}t}\times\underline{z}$

Solution

(a) $\underline{w}\cdot\underline{z} = (3t\underline{i} - t^2\underline{j})\cdot(2t^2\underline{i} + 3\underline{j}) = 6t^3 - 3t^2$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\underline{w}\cdot\underline{z}) = 18t^2 - 6t$$

Also

$$\frac{\mathrm{d}\underline{w}}{\mathrm{d}t} = 3\underline{i} - 2t\underline{j} \qquad \frac{\mathrm{d}\underline{z}}{\mathrm{d}t} = 4t\underline{i}$$

so

$$\begin{aligned}\underline{w}\cdot\frac{\mathrm{d}\underline{z}}{\mathrm{d}t} + \underline{z}\cdot\frac{\mathrm{d}\underline{w}}{\mathrm{d}t} &= (3t\underline{i} - t^2\underline{j})\cdot(4t\underline{i}) + (2t^2\underline{i} + 3\underline{j})\cdot(3\underline{i} - 2t\underline{j}) \\ &= 12t^2 + 6t^2 - 6t = 18t^2 - 6t\end{aligned}$$

We have verified $\frac{\mathrm{d}}{\mathrm{d}t}(\underline{w}\cdot\underline{z}) = \underline{w}\cdot\frac{\mathrm{d}\underline{z}}{\mathrm{d}t} + \frac{\mathrm{d}\underline{w}}{\mathrm{d}t}\cdot\underline{z}$

(b) $\underline{w}\times\underline{z} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3t & -t^2 & 0 \\ 2t^2 & 3 & 0 \end{vmatrix} = (9t + 2t^4)\underline{k}$ implying $\frac{\mathrm{d}}{\mathrm{d}t}(\underline{w}\times\underline{z}) = (9 + 8t^3)\underline{k}$

Also,

$$\begin{aligned}\underline{w}\times\frac{\mathrm{d}\underline{z}}{\mathrm{d}t} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3t & -t^2 & 0 \\ 4t & 0 & 0 \end{vmatrix} \\ &= 4t^3\underline{k}\end{aligned}$$

Solution (contd.)

$$\frac{d\underline{w}}{dt} \times \underline{z} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -2t & 0 \\ 2t^2 & 3 & 0 \end{vmatrix}$$
$$= (9 + 4t^3)\underline{k}$$

and so,

$$\underline{w} \times \frac{d\underline{z}}{dt} + \frac{d\underline{w}}{dt} \times \underline{z} = 4t^3\underline{k} + (9 + 4t^3)\underline{k} = (9 + 8t^3)\underline{k} = \frac{d}{dt}(\underline{w} \times \underline{z})$$

as required.

Exercises

1. If $r = 3t\underline{i} + 2t^2\underline{j} + t^3\underline{k}$, find

   (a) $\frac{dr}{dt}$ (b) $\frac{d^2r}{dt^2}$

2. Given $\underline{B} = te^{-t}\ \underline{i} + \cos t\ \underline{j}$ find

   (a) $\frac{d\underline{B}}{dt}$ (b) $\frac{d^2\underline{B}}{dt^2}$

3. If $\underline{r} = 4t^2\underline{i} + 2t\underline{j} - 7\underline{k}$ evaluate $\underline{r}$ and $\frac{d\underline{r}}{dt}$ when $t = 1$.

4. If $\underline{w} = t^3\underline{i} - 7t\underline{k}$, and $\underline{z} = (2 + t)\underline{i} + t^2\underline{j} - 2\underline{k}$

   (a) find $\underline{w} \cdot \underline{z}$, (b) find $\frac{d\underline{w}}{dt}$, (c) find $\frac{d\underline{z}}{dt}$, (d) show that $\frac{d}{dt}(\underline{w} \cdot \underline{z}) = \underline{w} \cdot \frac{d\underline{z}}{dt} + \frac{d\underline{w}}{dt} \cdot \underline{z}$

5. Given $\underline{r} = \sin t\ \underline{i} + \cos t\ \underline{j}$ find

   (a) $\dot{\underline{r}}$, (b) $\ddot{\underline{r}}$, (c) $|\underline{r}|$

   Show also that the position vector and velocity vector are perpendicular.

Answers

1. (a) $3\underline{i} + 4t\underline{j} + 3t^2\underline{k}$ (b) $4\underline{j} + 6t\underline{k}$
2. (a) $(-te^{-t} + e^{-t})\underline{i} - \sin t\underline{j}$ (b) $e^{-t}(t - 2)\underline{i} - \cos t\underline{j}$
3. $4\underline{i} + 2\underline{j} - 7\underline{k}$, $8\underline{i} + 2\underline{j}$
4. (a) $t(t^3 + 2t^2 + 14)$ (b) $3t^2\underline{i} - 7\underline{k}$ (c) $\underline{i} + 2t\underline{j}$
5. (a) $\cos t\underline{i} - \sin t\underline{j}$ (b) $-\sin t\underline{i} - \cos t\underline{j}$ (c) 1.