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integration

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Learning outcomes

In this workbook you will learn about integration and about some of the common techniques employed to obtain integrals. You will learn that integration is the inverse operation to differentiation and will also appreciate the distinction between a definite and an indefinite integral. You will understand how a definite integral is related to the area under a curve. You will understand how to use the technique of integration by parts to obtain integrals involving the product of functions. You will also learn how to use partial fractions and trigonometric identities in integration.

Time allocation

You are expected to spend approximately eight hours of independent study on the material presented in this workbook. However, depending upon your ability to concentrate and on your previous experience with certain mathematical topics this time may vary considerably.

Basic Concepts of Integration

13.1

Introduction

When a function $f(x)$ is known we can differentiate it to obtain its derivative $\frac{df}{dx}$. The reverse process is to obtain the function $f(x)$ from knowledge of its derivative. This process is called **integration**. Applications of integration are numerous and some of these will be explored in subsequent sections. For now, what is important is that you practise basic techniques and learn a variety of methods for integrating functions.

Prerequisites

Before starting this Section you should ...

① thoroughly understand the various techniques of differentiation

Learning Outcomes

After completing this Section you should be able to ...

✓ find some simple integrals by reversing the process of differentiation

✓ use a table of integrals

✓ explain the need for a constant of integration when finding indefinite integrals

✓ use the rules for finding integrals of sums of functions and constant multiples of functions

1. Integration as Differentiation in Reverse

Suppose we differentiate the function $y = x^2$. We obtain $\frac{dy}{dx} = 2x$. Integration reverses this process and we say that the integral of $2x$ is x^2. Pictorially we can regard this as shown in Figure 1:

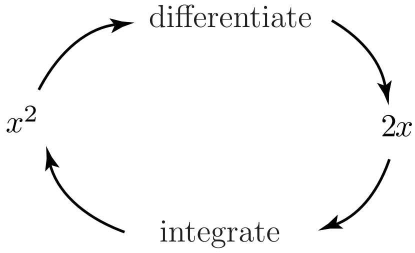


Figure 1.

The situation is just a little more complicated because there are lots of functions we can differentiate to give $2x$. Here are some of them:

$$x^2 + 4, \qquad x^2 - 15, \qquad x^2 + 0.5$$

Write down some more functions which have derivative $2x$.

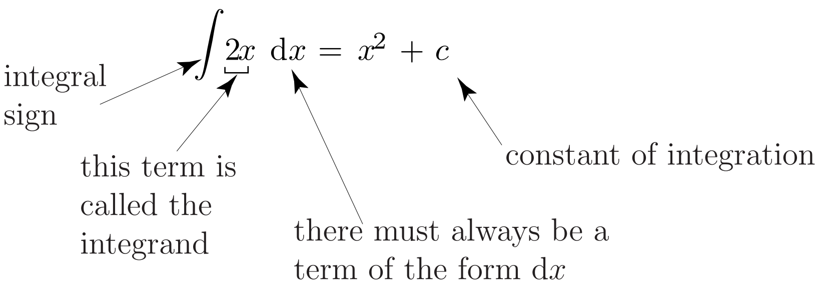
Your solution

e.g. $x^2 - 7$, $x^2 + 0.1$

All these functions have the same derivative, $2x$, because when we differentiate the constant term we obtain zero. Consequently, when we reverse the process, we have no idea what the original constant term might have been. So we include in our answer an unknown constant, c say, called the **constant of integration**. We state that the integral of $2x$ is $x^2 + c$.

When we want to differentiate a function, $y(x)$, we use the notation $\frac{d}{dx}$ as an instruction to differentiate, and write $\frac{d}{dx}\big(y(x)\big)$. In a similar way, when we want to integrate a function we use a special notation: $\int y(x)\,dx$.

The symbol for integration, $\int$, is known as an **integral sign**. To integrate $2x$ we write



Note that along with the integral sign there is a term of the form $\mathrm{d}x$, which must always be written, and which indicates the variable involved, in this case x. We say that $2x$ is being *integrated with respect to* x. The function being integrated is called the **integrand**. Technically, integrals of this sort are called **indefinite integrals**, to distinguish them from definite integrals which are dealt with subsequently. When you find an indefinite integral your answer should always contain a constant of integration.

Exercises

1 a) Write down the derivatives of each of:

$$x^3, \qquad x^3 + 17, \qquad x^3 - 21$$

b) Deduce that $\int 3x^2 \mathrm{d}x = x^3 + c$.

2. What is meant by the term 'integrand'?
3. Explain why, when finding an indefinite integral, a constant of integration is always needed.

2. A Table of Integrals

We could use a table of derivatives to find integrals, but the more common ones are usually found in a 'Table of Integrals' such as that shown below. You could check the entries in this table using your knowledge of differentiation. Try this for yourself.

Table of integrals

function $f(x)$	indefinite integral $\int f(x)\mathrm{d}x$
constant, k	$kx + c$
x	$\frac{1}{2}x^2 + c$
x^2	$\frac{1}{3}x^3 + c$
x^n	$\dfrac{x^{n+1}}{n+1} + c, \quad n \neq -1$
x^{-1} (or $\dfrac{1}{x}$)	$\ln\|x\| + c$
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$
$\cos kx$	$\dfrac{1}{k}\sin kx + c$
$\sin kx$	$-\dfrac{1}{k}\cos kx + c$
$\tan kx$	$\dfrac{1}{k}\ln\|\sec kx\|$+c
e^x	$\mathrm{e}^x + c$
e^{-x}	$-\mathrm{e}^{-x} + c$
e^{kx}	$\dfrac{1}{k}\mathrm{e}^{kx} + c$

When dealing with the trigonometric functions the variable x must always be measured in radians and not degrees. Note that the fourth entry in the table is valid for any value of n, positive, negative, or fractional, *except* $n = -1$. When $n = -1$ use the fifth entry in the table.

Example Use the table above to find the indefinite integral of x^7: that is, find $\int x^7 \mathrm{d}x$

Solution

From the table note that $\displaystyle\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + c$. In words, this states that to integrate a power of x, increase the power by 1, and then divide the result by the new power. With $n = 7$ we find

$$\int x^7 \mathrm{d}x = \frac{1}{8}x^8 + c$$

Example Find the indefinite integral of $\cos 5x$: that is, find $\int \cos 5x \,\mathrm{d}x$

Solution

From the table note that

$$\int \cos kx \,\mathrm{d}x = \frac{\sin kx}{k} + c$$

With $k = 5$ we find

$$\int \cos 5x \,\mathrm{d}x = \frac{1}{5}\sin 5x + c$$

In the table the independent variable is always given as x. However, with a little imagination you will be able to use it when other independent variables are involved.

Example Find $\int \cos 5t \,\mathrm{d}t$

Solution

We integrated $\cos 5x$ in the previous example. Now the independent variable is t, so simply use the table and read every x as a t. With $k = 5$ we find

$$\int \cos 5t \,\mathrm{d}t = \frac{1}{5}\sin 5t + c$$

It follows immediately that, for example,

$$\int \cos 5\omega \,\mathrm{d}\omega = \frac{1}{5}\sin 5\omega + c, \qquad \int \cos 5u \,\mathrm{d}u = \frac{1}{5}\sin 5u + c$$

and so on.

Example Find the indefinite integral of $\frac{1}{x}$: that is, find $\int \frac{1}{x}\,dx$

Solution

This integral deserves special mention. You may be tempted to try to write the integrand as x^{-1} and use the fourth row of the Table. However, the formula $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ is not valid when $n = -1$ as the Table makes clear. This is because we can never divide by zero. Look to the fifth entry of the Table and you will see $\int x^{-1}\,dx = \ln|x| + c$.

Example Find $\int 12\,dx$

Solution

In this example we are integrating a constant, 12. Using the table we find

$$\int 12\,dx = 12x + c$$

Note that $\displaystyle\int 12dt$ would be $12t + c$.

Find $\int t^4\,dt$

Your solution

$\int t^4 dt = \frac{1}{5}t^5 + c.$

Find $\int \frac{1}{x^5}dx$

Your solution

Use the laws of indices to write the integrand as x^{-5} and then use the Table.

$-\frac{1}{4}x^{-4} + c = -\frac{1}{4x^4} + c.$

Find $\int e^{-2x} dx$.

Your solution

Use the entry in the table for integrating e^{kx}.

With $k = -2$, we have $\int e^{-2x} dx = -\frac{1}{2}e^{-2x} + c.$

Exercises

1. Integrate each of the following functions:
   a) x^9, b) $x^{1/2}$, c) x^{-3}, d) $1/x^4$, e) 4, f) $\sqrt{x}$, g) e^{4x}
2. Find a) $\int t^2\, dt$, b) $\int 6\, dt$, c) $\int \sin 3t\, dt$, d) $\int e^{7t}\, dt$.
3. Find $\int e^t\, dt$.

Answers 1 a) $\frac{1}{10}x^{10} + c$, b) $\frac{2}{3}x^{3/2} + c$, c) $-\frac{1}{2}x^{-2} + c$, d) $-\frac{1}{3}x^{-3} + c$, e) $4x + c$,
f) same as b), g) $\frac{1}{4}e^{4x} + c$

2. a) $\frac{1}{3}t^3 + c$, b) $6t + c$, c) $-\frac{1}{3}\cos 3t + c$, d) $\frac{1}{7}e^{7t} + c$

3. $e^t + c$

3. Some Rules of Integration

To enable us to find integrals of a wider range of functions than those normally given in a table of integrals we can make use of the following rules.

The integral of $k\,f(x)$ where k is a constant

A constant factor in an integral can be moved outside the integral sign as follows:

Key Point

$$\int k\, f(x)\, dx = k \int f(x) dx$$

Example Find the indefinite integral of $11x^2$: that is, find $\int 11x^2\, dx$

Solution

$$\int 11x^2\, dx = 11 \int x^2\, dx = 11\left(\frac{x^3}{3} + c\right) = \frac{11x^3}{3} + K$$

where K is a constant.

Example Find the indefinite integral of $-5\cos x$; that is, find $\int -5\cos x\, dx$

Solution

$$\int -5\cos x\, dx = -5 \int \cos x\, dx = -5\,(\sin x + c) = -5\sin x + K$$

where K is a constant.

The integral of $f(x) + g(x)$ or of $f(x) - g(x)$

When we wish to integrate the sum or difference of two functions, we integrate each term separately as follows:

Key Point

$$\int [\, f(x) + g(x)\,]\, dx = \int f(x)\, dx + \int g(x)\, dx$$

$$\int [\, f(x) - g(x)\,]\, dx = \int f(x)\, dx - \int g(x)\, dx$$

Example Find $\int (x^3 + \sin x)\mathrm{d}x$

Solution

$$\int (x^3 + \sin x)\mathrm{d}x = \int x^3\,\mathrm{d}x + \int \sin x\,\mathrm{d}x = \frac{1}{4}x^4 - \cos x + c$$

Note that only a single constant of integration is needed.

Find $\int (3t^4 + \sqrt{t})\,\mathrm{d}t$

Your solution

You will need to use both of the rules to deal with this integral.

$\frac{3}{5}t^5 + \frac{2}{3}t^{3/2} + c$

The hyperbolic sine and cosine functions, $\sinh x$ and $\cosh x$ are defined as follows:

$$\sinh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} \qquad \cosh x = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}$$

Note that they are simply combinations of the exponential functions e^x and e^{-x}.

Find the indefinite integrals of $\sinh x$ and $\cosh x$.

Your solution

$$\int \sinh x\,\mathrm{d}x = \int \left(\frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}\right)\mathrm{d}x =$$

$$\int \cosh x\,\mathrm{d}x = \int \left(\frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}\right)\mathrm{d}x =$$

$$\int \sinh x \, dx = \frac{1}{2}\int e^x dx - \frac{1}{2}\int e^{-x} dx = \frac{1}{2}e^x + \frac{1}{2}e^{-x} + c = \frac{1}{2}\left(e^x + e^{-x}\right) + c = \cosh x + c.$$

Similarly $\int \cosh x \, dx = \sinh x + c$.

Further rules for finding more complicated integrals are dealt with in subsequent sections.

Exercises

1. Find $\int (2x - e^x) dx$
2. Find $\int 3e^{2x} dx$
3. Find $\int \frac{1}{3}(x + \cos 2x) \, dx$
4. Find $\int 7x^{-2} dx$
5. Find $\int (x+3)^2 dx$, (be careful!)

Answers 1. $x^2 - e^x + c$ 2. $\frac{3}{2}e^{2x} + c$ 3. $\frac{1}{6}x^2 + \frac{1}{6}\sin 2x + c$ 4. $-\frac{7}{x} + c$ 5. $\frac{1}{3}x^3 + 3x^2 + 9x + c$