Integration by Parts

13.4

Introduction

Integration by Parts is a technique for integrating products of functions. In this section you will learn to recognise when it is appropriate to use the technique and have the opportunity to practise using it for finding both definite and indefinite integrals.

Prerequisites

Before starting this Section you should . . .

① understand what is meant by definite and indefinite integrals

② be able to use a table of integrals

③ be able to differentiate and integrate a range of common functions

Learning Outcomes

After completing this Section you should be able to . . .

✓ decide when it is appropriate to use the method known as integration by parts

✓ apply the formula for integration by parts to definite and indefinite integrals

✓ apply the formula repeatedly if appropriate

1. Indefinite Integration

The technique known as **integration by parts** is used to integrate a product of two functions, for example

$$\int e^{2x} \sin 3x \, dx \qquad \text{and} \qquad \int_0^1 x^3 e^{-2x} \, dx$$

Note that in the first example, the integrand is the product of the functions e^{2x} and $\sin 3x$, and in the second example the integrand is the product of the functions x^3 and e^{-2x}. Note also that we can change the order of the terms in the product if we wish and write

$$\int (\sin 3x) \, e^{2x} dx \qquad \text{and} \qquad \int_0^1 e^{-2x} \, x^3 \, dx$$

What you must never do is integrate each term in the product separately - the integral of a product is not the product of the separate integrals. However, it is often possible to find integrals involving products using the method of integration by parts - you can think of this as a *product rule* for integrals.

The integration by parts formula states:

Key Point

For indefinite integrals: given functions $f(x)$ and $g(x)$

$$\int f \cdot g \, dx = f \cdot \int g \, dx - \int \left(\frac{df}{dx} \cdot \int g dx \right) dx$$

Study the formula carefully and note the following observations. Firstly, to apply the formula we must be able to differentiate the function f to find $\frac{df}{dx}$. and we must be able to integrate the function, g.

Secondly the formula replaces one integral, the one on the left, with a different integral, that on the far right. The intention is that the latter, whilst it may look more complicated in the formula above, is simpler to evaluate.

Consider the following example:

Example Find the integral of the product of x with $\sin x$; that is, find $\int x \sin x \, dx$.

Solution

Compare the required integral with the formula for integration by parts: we choose

$$f = x \qquad \text{and} \qquad g = \sin x$$

It follows that

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 1 \qquad \text{and} \qquad \int g \, \mathrm{d}x = \int \sin x \, \mathrm{d}x = -\cos x$$

(When integrating g there is no need to worry about a constant of integration. When you become confident with the method, you may like to think about why this is the case.) Applying the formula we obtain

$$\begin{aligned} \int x \sin x \, \mathrm{d}x &= f \cdot \int g \, \mathrm{d}x - \int \left(\frac{\mathrm{d}f}{\mathrm{d}x} \cdot \int g \mathrm{d}x \right) \mathrm{d}x \\ &= x(-\cos x) - \int 1(-\cos x) \, \mathrm{d}x \\ &= -x \cos x + \int \cos x \, \mathrm{d}x \\ &= -x \cos x + \sin x + c \end{aligned}$$

We have used the method of integration by parts to determine the integral of a product of two functions.

Find $\int (5x + 1) \cos 2x \, \mathrm{d}x$.

Your solution

Let $f = 5x + 1$ and $g = \cos 2x$. Now calculate $\frac{\mathrm{d}f}{\mathrm{d}x}$ and $\int g \, \mathrm{d}x$.

$\frac{\mathrm{d}f}{\mathrm{d}x} = 5$ and $\int \cos 2x \, \mathrm{d}x = \frac{1}{2} \sin 2x$.

Your solution

Substitute these results into the formula for integration by parts:

$\frac{1}{2}(5x + 1) \sin 2x + \frac{5}{4} \cos 2x + c$

Exercises

In some questions it may be necessary to apply the formula more than once.

1. Find a) $\int x\sin(2x)\mathrm{d}x$, b) $\int t\mathrm{e}^{3t}\mathrm{d}t$, c) $\int x\cos x\,\mathrm{d}x$.
2. Find $\int (x+3)\sin x\,\mathrm{d}x$.
3. By writing $\ln x$ as $1\times\ln x$ find $\int \ln x\,\mathrm{d}x$.
4. Find a) $\int \tan^{-1} x\,\mathrm{d}x$, b) $\int -7x\cos 3x\,\mathrm{d}x$, c) $\int 5x^2\mathrm{e}^{3x}\mathrm{d}x$, d) $\int (x+1)\mathrm{e}^x\mathrm{d}x$

In questions 5, k is a constant.

5. Find a) $\int x\cos kx\,\mathrm{d}x$, b) $\int z^2\cos kz\,dz$.
6. Find a) $\int t\mathrm{e}^{-st}dt$ b) Find $\int t^2\mathrm{e}^{-st}dt$ where s is a constant.

6(a) $\dfrac{-\mathrm{e}^{-st}(st+1)}{s^2}+c.$ 6(b) $\dfrac{-\mathrm{e}^{-st}(s^2t^2+2st+2)}{s^3}+c.$

5. a) $\dfrac{\cos kx}{k^2}+\dfrac{x\sin kx}{k}+c$, b) $\dfrac{2z\cos kz}{k^2}+\dfrac{z^2\sin kz}{k}-\dfrac{2\sin kz}{k^3}+c.$

4. a) $x\tan^{-1}x-\frac{1}{2}\ln(x^2+1)+c$, b) $-\frac{7}{9}\cos 3x-\frac{7}{3}x\sin 3x+c$, c) $\frac{5}{27}\mathrm{e}^{3x}(9x^2-6x+2)+c$, d) $x\mathrm{e}^x+c.$

3. $x\ln x-x+c.$

2. $-(x+3)\cos x+\sin x+c.$

1. a) $\frac{1}{4}\sin 2x-\frac{1}{2}x\cos 2x+c$, b) $\mathrm{e}^{3t}(\frac{1}{3}t-\frac{1}{9})+c$, c) $\cos x+x\sin x+c$

2. Definite Integrals

When dealing with definite integrals the relevant formula is as follows:

Key Point

For definite integrals: given functions $f(x)$ and $g(x)$

$$\int_a^b f\cdot g\,\mathrm{d}x=\left[f\cdot\int g\,\mathrm{d}x\right]_a^b-\int_a^b\left(\frac{\mathrm{d}f}{\mathrm{d}x}\cdot\int g\mathrm{d}x\right)\mathrm{d}x$$

Example Find the definite integral of x with e^x from 0 to 2; that is, find $\int_0^2 x\mathrm{e}^x\mathrm{d}x$.

Solution

We let $f = x$ and $g = \mathrm{e}^x$. Then $\frac{\mathrm{d}f}{\mathrm{d}x} = 1$ and $\int g\,\mathrm{d}x = \mathrm{e}^x$. Using the formula for integration by parts we obtain

$$\begin{aligned}\int_0^2 x\mathrm{e}^x\mathrm{d}x &= [x\mathrm{e}^x]_0^2 - \int_0^2 1.\mathrm{e}^x\mathrm{d}x \\ &= 2\mathrm{e}^2 - [\mathrm{e}^x]_0^2 \\ &= 2\mathrm{e}^2 - [\mathrm{e}^2 - 1] = \mathrm{e}^2 + 1 \qquad \text{(or 8.389 to 3 d.p.)}\end{aligned}$$

Sometimes it is necessary to apply the formula more than once as the next example shows.

Example Find the definite integral of $x^2\mathrm{e}^x$ from 0 to 2; that is, find $\int_0^2 x^2\mathrm{e}^x\mathrm{d}x$.

Solution

We let

$$f = x^2 \qquad \text{and} \qquad g = \mathrm{e}^x$$

Then

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 2x \qquad \text{and} \qquad \int g\,\mathrm{d}x = \mathrm{e}^x$$

Using the formula for integration by parts we find

$$\begin{aligned}\int_0^2 x^2\mathrm{e}^x\mathrm{d}x &= [x^2\mathrm{e}^x]_0^2 - \int_0^2 2x\mathrm{e}^x\mathrm{d}x \\ &= 4\mathrm{e}^2 - 2\int_0^2 x\mathrm{e}^x\mathrm{d}x\end{aligned}$$

The remaining integral must be integrated by parts also but we have just done this in the example above. So

$$\int_0^2 x^2\mathrm{e}^x\mathrm{d}x = 4\mathrm{e}^2 - 2[\mathrm{e}^2 + 1] = 2\mathrm{e}^2 - 2 = 12.778 \qquad \text{(3 d.p.)}$$

Find $\int_0^{\pi/4}(4 - 3x)\sin x\,\mathrm{d}x$.

Your solution

What are your choices for f, g?

$$f(x) = \qquad\qquad\qquad\qquad g(x) =$$

Now complete the integral

$$\int_0^{\pi/4} (4-3x)\sin x \, \mathrm{d}x =$$

Take $f = 4 - 3x$ and $g = \sin x$. This gives

$$\begin{aligned} \int_0^{\pi/4} (4-3x)\sin x \, \mathrm{d}x &= [(4-3x)(-\cos x)]_0^{\pi/4} - 3\int_0^{\pi/4} \cos x \, \mathrm{d}x \\ &= [(4-3x)(-\cos x)]_0^{\pi/4} - 3[\sin x]_0^{\pi/4} \\ &= 0.716 \end{aligned}$$

Exercises

1. Evaluate the following: a) $\int_0^1 x\cos 2x \, \mathrm{d}x$, b) $\int_0^{\pi/2} x\sin 2x \, \mathrm{d}x$, c) $\int_{-1}^1 t\mathrm{e}^{2t}\mathrm{d}t$
2. Find $\int_1^2 (x+2)\sin x \, \mathrm{d}x$
3. Find $\int_0^1 (x^2 - 3x + 1)\mathrm{e}^x \mathrm{d}x$

Answers 1. a) 0.1006, b) $\pi/4 = 0.7854$, c) 1.9488. 2. 3.3533, 3. -0.5634.