Integration of Trigonometric Functions

13.6

Introduction

Integrals involving trigonometric functions are commonplace in engineering mathematics. This is especially true when modelling waves, and alternating current circuits. When the root-mean-square (rms) value of a waveform, or signal is to be calculated, you will often find this results in an integral of the form

$$\int \sin^2 t \, dt$$

In this Section you will learn how such integrals can be evaluated.

Prerequisites

Before starting this Section you should ...

① be able to find a number of simple definite and indefinite integrals

② be able to use a table of integrals

③ be familiar with standard trigonometric identities

Learning Outcomes

After completing this Section you should be able to ...

✓ use trigonometric identities to write integrands in alternative forms to enable them to be integrated

1. Integration of Trigonometric Functions

Simple integrals involving trigonometric functions have already been dealt with in Section 13.1. See what you can remember:

Write down the following integrals:

a) $\int \sin x \, \mathrm{d}x$, b) $\int \cos x \, \mathrm{d}x$, c) $\int \sin 2x \, \mathrm{d}x$, d) $\int \cos 2x \, \mathrm{d}x$

Your solution

a) $-\cos x + c$, b) $\sin x + c$, c) $-\frac{1}{2}\cos 2x + c$, d) $\frac{1}{2}\sin 2x + c$.

The basic rules from which these results can be derived are summarised here:

Key Point

$$\int \sin kx \, \mathrm{d}x = -\frac{\cos kx}{k} + c \qquad \int \cos kx \, \mathrm{d}x = \frac{\sin kx}{k} + c$$

In engineering applications it is often necessary to integrate functions involving powers of the trigonometric functions such as

$$\int \sin^2 x \, \mathrm{d}x \qquad \text{or} \qquad \int \cos^2 \omega t \, \mathrm{d}t$$

Note that these integrals cannot be obtained directly from the formulas in the Key Point above. However by making use of trigonometric identities the integrands can be re-written in an alternative form. It is often not clear which identities are useful and each case needs to be considered individually. Experience and practice are essential. Work through the following Guided Exercise.

Use the trigonometric identity

$$\sin^2 \theta = \tfrac{1}{2}(1 - \cos 2\theta)$$

to express the integral $\int \sin^2 x \, \mathrm{d}x$ in an alternative form.

Your solution

The integral can be written $\int \frac{1}{2}(1 - \cos 2x)\mathrm{d}x$.

Note from the last exercise that the trigonometric identity was used to convert a power of $\sin x$ into a function involving $\cos 2x$ which can be integrated directly using the Key Point above.

Now find the indefinite integral $\int \sin^2 x \, \mathrm{d}x$.

Your solution

$\frac{1}{2}\left(x - \frac{1}{2}\sin 2x + c\right) = \frac{1}{2}x - \frac{1}{4}\sin 2x + K$ where $K = c/2$.

Use the trigonometric identity $\sin 2x = 2\sin x \cos x$ to find $\int \sin x \cos x \, \mathrm{d}x$

Your solution

The integrand can be written as $\frac{1}{2}\sin 2x$ whose indefinite integral is $-\frac{1}{4}\cos 2x + c$

Using the result of the previous example write down the value of

$$\int_0^{2\pi} \sin x \cos x \; \mathrm{d}x$$

Your solution

$$\begin{aligned}\int_0^{2\pi} \sin x \cos x \, \mathrm{d}x &= \left[-\frac{1}{4}\cos 2x + c\right]_0^{2\pi} \\ &= -\frac{1}{4}\cos 4\pi + \frac{1}{4}\cos 0 \\ &= -\frac{1}{4} + \frac{1}{4} = 0\end{aligned}$$

This result is one example of what are called **orthogonality relations**.

2. Orthogonality Relations

In general two functions $f(x), g(x)$ are said to be **orthogonal** to each other over an interval $a \leq x \leq b$ if

$$\int_a^b f(x)g(x)\, \mathrm{d}x = 0$$

It follows from the previous example that $\sin x$ and $\cos x$ are orthogonal to each other over the interval $0 \leq x \leq 2\pi$ or indeed any other interval $\alpha \leq x \leq \alpha + 2\pi$ (e.g. $\pi/2 \leq x \leq 5\pi$ or $-\pi \leq x \leq \pi$).

More generally there is a whole set of orthogonality relations involving these trigonometric functions on intervals of length 2π (i.e. over one period of both $\sin x$ and $\cos x$). These relations are useful in connection with a widely used technique in engineering, known as **Fourier analysis** where we represent periodic functions in terms of an infinite series of sines and cosines called a Fourier series.

We shall demonstrate the orthogonality property

$$I_{mn} = \int_0^{2\pi} \sin mx \sin nx \, \mathrm{d}x = 0$$

where m and n are integers such that $m \neq n$.

The secret is to use a trigonometric identity to convert the integrand into a form that can be readily integrated.

You may recall the identity

$$\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$$

It follows, putting $A = mx$ and $B = nx$ that

$$\begin{aligned}I_{mn} &= \frac{1}{2}\int_0^{2\pi} [\cos(m-n)x - \cos(m+n)x]\, \mathrm{d}x \\ &= \frac{1}{2}\left[\frac{\sin(m-n)x}{(m-n)} - \frac{\sin(m+n)x}{(m+n)}\right]_0^{2\pi} \\ &= 0\end{aligned}$$

because $(m-n)$ and $(m+n)$ will be integers and $\sin(\text{integer}) \times 2\pi) = 0$. Also of course $\sin 0 = 0$.

Why does the case $m = n$ have to be excluded from the analysis?

The corresponding orthogonality relation for cosines

$$J_{mn} = \int_0^{2\pi} \cos mx \cos nx \; \mathrm{d}x = 0$$

follows by use of a similar identity to that just used. Here again m and n are integers such that $m \neq n$.

Use the identity

$$\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$$

to show that

$$K_{mn} = \int_0^{2\pi} \sin mx \cos nx \; \mathrm{d}x = 0 \qquad m \text{ and } n \text{ integers, } m \neq n.$$

Your solution

We have, by the given identity,

$$\begin{aligned} K_{mn} &= \frac{1}{2}\int_0^{2\pi} [\sin(m+n)x + \sin(m-n)x] \; \mathrm{d}x \\ &= \frac{1}{2}\left[-\frac{\cos(m+n)x}{(m+n)} - \frac{\cos(m-n)x}{(m-n)}\right]_0^{2\pi} \\ &= -\frac{1}{2}\left[\frac{\cos(m+n)2\pi - 1}{(m+n)} + \frac{\cos(m-n)2\pi - 1}{(m-n)}\right] \\ &= 0 \end{aligned}$$

(recalling that cos((integer) $\times\, 2\pi$) = 1)

Finally show that the orthogonality relation K_{mn} also holds if $m = n$. Hint: You will need to use a different trigonometric identity.

Your solution

$$K_{mn} = \int_0^{2\pi} \sin mx \cos mx \ \mathrm{d}x$$

Putting $m = n$, and then using the identity $\sin 2A = 2 \sin A \cos A$ we get

$$\begin{aligned} K_{mm} &= \int_0^{2\pi} \sin mx \cos mx \ \mathrm{d}x \\ &= \frac{1}{2}\int_0^{2\pi} \sin 2mx \ \mathrm{d}x \\ &= \frac{1}{2}\left[-\frac{\cos 2mx}{2m}\right]_0^{2\pi} \\ &= -\frac{1}{4m}(\cos 4m\pi - \cos 0) \\ &= -\frac{1}{4m}(1 - 1) = 0 \end{aligned}$$

Note that the particular case $m = n = 1$ was considered earlier in this Section.

3. Reduction Formulae

You have seen earlier in this Workbook how to integrate $\sin x$ and $\sin^2 x$ (which is $\sin x$ multiplied by itself). Applications sometimes arise which involve integrating higher powers of $\sin x$ or $\cos x$. It is possible, as we now show, to obtain a *reduction formula* to aid in this task.

So consider

$$I_n = \int \sin^n(x) \ \mathrm{d}x$$

Write down the precise integrals represented by I_2, I_3, I_{10}

Your solution

$I_2 = \int \sin^2 x \; \mathrm{d}x \qquad I_3 = \int \sin^3 x \; \mathrm{d}x \qquad I_{10} = \int \sin^{10} x \; \mathrm{d}x$

To obtain a reduction formula for I_n we write

$$\sin^n x = \sin^{n-1}(x) \sin x$$

and use integration by parts.

In the notation used earlier in this Workbook put $f = \sin^{n-1} x$ and $g = \sin x$ and evaluate $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ and $\displaystyle\int g \; \mathrm{d}x$.

Your solution

We have

$$\frac{\mathrm{d}f}{\mathrm{d}x} = (n-1)\sin^{n-2} x \; \cos x \quad \text{(using the chain rule of differentiation)}$$

$$\int g \; \mathrm{d}x = \int \sin x \; \mathrm{d}x = -\cos x$$

Now use the integration by parts formula on $\displaystyle\int \sin^{n-1} x \sin x \; \mathrm{d}x$. (See earlier in the Workbook for the parts formula if necessary). Do not attempt to evaluate the second integral that you obtain.

Your solution

$$\begin{aligned} \int \sin^{n-1} x \sin x \; \mathrm{d}x &= \sin^{n-1}(x) \int g \; \mathrm{d}x - \int \frac{\mathrm{d}f}{\mathrm{d}x} \int g \; \mathrm{d}x \\ &= \sin^{n-1}(x)(-\cos x) + (n-1) \int \sin^{n-2} x \cos^2 x \; \mathrm{d}x \end{aligned}$$

Putting $\cos^2 x = 1 - \sin^2 x$ in the integral on the right-hand side, this integral becomes:

$$\int \sin^{n-2}(x)\ \mathrm{d}x - \int \sin^n(x)\ \mathrm{d}x$$

so finally

$$I_n = \int \sin^{n-1}(x)\sin x\ \mathrm{d}x = \sin^{n-1}(x)(-\cos x)+(n-1)\int \sin^{n-2}(x)\ \mathrm{d}x-(n-1)\int \sin^n(x)\ \mathrm{d}x$$

or

$$I_n = -\sin^{n-1}(x)\cos x + (n-1)I_{n-2} - (n-1)I_n$$

from which

$$I_n = -\frac{1}{n}\sin^{n-1}(x)\cos x + \frac{n-1}{n}I_{n-2} \qquad (*)$$

This is our reduction formula for I_n. It enables us, for example, to evaluate I_6 in terms of I_4, then I_4 in terms of I_2 and indeed I_2 in terms of I_0 where

$$I_0 = \int \sin^0 x\ \mathrm{d}x = \int 1\ \mathrm{d}x = x.$$

Use the reduction formula with $n = 2$ to calculate I_2.

Your solution

$$\begin{aligned} I_2 &= -\frac{1}{2}[\sin x\cos x] + \frac{1}{2}I_0 \\ &= -\frac{1}{2}[\frac{1}{2}\sin 2x] + \frac{x}{2} \end{aligned}$$

i.e. $\displaystyle\int \sin^2 x\ \mathrm{d}x = -\frac{1}{4}\sin 2x + \frac{x}{2}$

as obtained earlier by a different technique.

(NOTE: here and elsewhere we are omitting the constant of integration.)

Use the reduction formula $(*)$ to obtain $I_6 = \displaystyle\int \sin^6 x\ \mathrm{d}x$. Firstly obtain I_6 in terms of I_4, then I_4 in terms of I_2. (You evaluated I_2 in the previous exercise.)

Your solution

Using $(*)$ with $n = 6$

$$I_6 = -\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}I_4$$

Then, by $(*)$ with $n = 4$

$$I_4 = -\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}I_2$$

Now substitute for I_2 from the previous exercise to obtain I_4 and hence I_6.

Your solution

$$I_4 = -\tfrac{1}{4}\sin^3 x\cos x - \tfrac{3}{16}\sin 2x + \tfrac{3}{8}x$$

$$\therefore \quad I_6 = -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x - \frac{5}{32}\sin 2x + \frac{5}{16}x$$

Definite integrals can also be readily evaluated using the reduction formula $(*)$. For example,

$$I_n = \int_0^{\pi/2} \sin nx \, \mathrm{d}x$$

so $I_{n-2} = \displaystyle\int_0^{\pi/2} \sin^{n-2} x \, \mathrm{d}x$

We obtain, immediately

$$I_n = \frac{1}{n}\left[-\sin^{n-1}(x)\cos x\right]_0^{\pi/2} + \frac{n-1}{n}I_{n-2}$$

or, since $\cos\frac{\pi}{2} = \sin 0 = 0$,

$$I_n = \frac{(n-1)}{n}I_{n-2}$$

This simple easy-to-use formula is well known and is called **Wallis' formula**.

If $I_n = \displaystyle\int_0^{\pi/2} \sin^n x \ \mathrm{d}x$ calculate I_1 and then use Wallis' formula, without further integration, to obtain I_3 and I_5.

Your solution

$I_1 = \int_0^{\pi/2} \sin x \ \mathrm{d}x = [-\cos x]_0^{\pi/2} = 1$

Then using Wallis' formula with $n = 3$ and $n = 5$ respectively

$$I_3 = \int_0^{\pi/2} \sin^3 x \ \mathrm{d}x = \frac{2}{3} I_1 = \frac{2}{3} \times 1 = \frac{2}{3}$$

$$I_5 = \int_0^{\pi/2} \sin^5 x \ \mathrm{d}x = \frac{4}{5} I_3 = \frac{4}{5} \times \frac{2}{3} = \frac{8}{15}$$

The total power P of an antenna is given by

$$P = \int_0^{\pi} \frac{\eta L^2 I^2 \pi}{4\lambda^2} \sin^3 \theta \ \mathrm{d}\theta$$

where η, λ, I are constants as is the length L of antenna. Using the reduction formula for $\int \sin^n x$, obtain P.

Your solution

> Ignoring the constants for the moment, consider
>
> $$I_3 = \int_0^\pi \sin^3\theta \ \mathrm{d}\theta.$$
>
> If $I_1 = \int_0^\pi \sin\theta \ \mathrm{d}\theta = [-\cos\theta]_0^\pi = 2$
>
> then by the reduction formula (*) with $n = 3$
>
> $$I_3 = \frac{1}{3}\left[-\sin^2 x \cos x\right]_0^\pi + \frac{2}{3}I_1 = \frac{2}{3}\times 2 = \frac{4}{3}$$
>
> Hence $\quad P = \dfrac{\eta L^2 I^2 \pi}{4\lambda^2}\displaystyle\int_0^\pi \sin^3\theta \ \mathrm{d}\theta = \dfrac{\eta L^2 I^2 \pi}{3\lambda^2}$

A similar reduction formula to (*) can be obtained for $\int \cos^n x \ \mathrm{d}x$ (see exercises).. In particular if

$$J_n = \int_0^{\pi/2} \cos^n x \ \mathrm{d}x \quad \text{then} \quad J_n = \tfrac{(n-1)}{n} J_{n-2}$$

i.e. Wallis' formula is the same for $\cos^n x$ as for $\sin^n x$.

4. Harder Trigonometric Integrals

The following seemingly innocent integrals are examples, important in engineering, of trigonometric integrals that **cannot** be evaluated as indefinite integrals:

(a) $\int \sin(x^2) \ \mathrm{d}x \qquad$ and $\qquad \int \cos(x^2) \ \mathrm{d}x$

These are called Fresnel integrals.

(b) $\int \dfrac{\sin x}{x} \ \mathrm{d}x$

This is called the Sine integral.

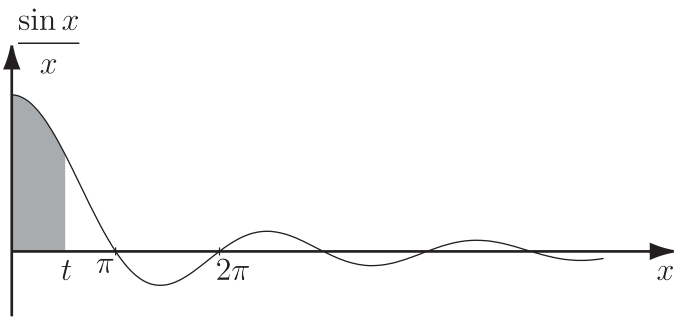
Definite integrals of this type, which are what normally arise in applications, have to be evaluated by approximate numerical methods.

Fresnel integrals with limits arise in wave and antenna theory and the Sine integral with limits in filter theory.

It is useful sometimes to be able to visualize the definite integral. For example consider

$$F(t) = \int_0^t \frac{\sin x}{x} \ \mathrm{d}x \qquad t > 0$$

Clearly, $F(0) = \displaystyle\int_0^0 \frac{\sin x}{x}\,\mathrm{d}x = 0$. Recall the graph of $\dfrac{\sin x}{x}$ against x, $x > 0$:

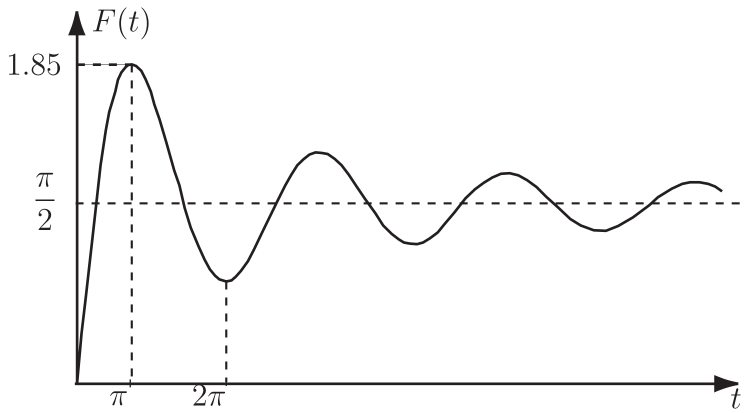


For any positive value of t, $F(t)$ is the shaded area shown (the area interpretation of a definite integral was covered earlier in this Workbook). As t increases from 0 to π, it follows that $F(t)$ increases from 0 to a maximum value

$$F(\pi) = \int_0^\pi \frac{\sin x}{x}\,\mathrm{d}x$$

whose value could be determined numerically (it is actually about 1.85). As t further increases from π to 2π the value of $F(t)$ will decrease to a local minimum at 2π because the $\dfrac{\sin x}{x}$ curve is below the x-axis between π and 2π.

Continuing to argue in this way we can obtain the shape of the $F(t)$ graph as follows: (can you see why the oscillations decrease in amplitude?)



The result

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}$$

is clear from the graph (you are not expected to know how this result is obtained). Such problems are dealt with in Workbook 31.

Exercises

You will need to refer to a Table of Trigonometric Identities to answer these questions.

1. Find a) $\int \cos^2 x \mathrm{d}x$ b) $\int_0^{\pi/2} \cos^2 t \mathrm{d}t$ c) $\int (\cos^2\theta + \sin^2\theta)\mathrm{d}\theta$
2. Use the identity $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$ to find $\int \sin 3x \cos 2x \mathrm{d}x$
3. Find $\int (1+\tan^2 x)\mathrm{d}x$.
4. The mean square value of a function $f(t)$ over the interval $t=a$ to $t=b$ is defined to be

$$\frac{1}{b-a}\int_a^b (f(t))^2 \mathrm{d}t$$

Find the mean square value of $f(t) = \sin t$ over the interval $t=0$ to $t=2\pi$.

5(a) Show that the reduction formula for $J_n = \displaystyle\int \cos^n x \ \mathrm{d}x$ is

$$J_n = \frac{1}{n}\cos^{n-1}(x)\sin x + \frac{(n-1)}{n}J_{n-2}$$

(b) Using the above reduction formula show that

$$\int \cos^5 x \ \mathrm{d}x = \frac{1}{5}\cos^4 x \sin x + \frac{4}{15}\cos^2 x \sin x + \frac{8}{15}\sin x$$

(c) Show that if

$$J_n = \int_0^{\pi/2} \cos^n x \ \mathrm{d}x \text{ then } J_n = \left(\frac{n-1}{n}\right) J_{n-2} \text{ (Wallis' formula).}$$

(d) Using Wallis' formula show that

$$\int_0^{\pi/2} \cos^6 x \ \mathrm{d}x = \frac{5}{32}\pi.$$

Answers 1. a) $\frac{1}{2}x + \frac{1}{4}\sin 2x + c$. b) $\pi/4$. c) $\theta + c$. 2. $-\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + c$. 3. $\tan x + c$. 4. $\frac{1}{2}$.