

Contents **14**

applications of

integration 1

1. Integration as the limit of a sum
2. The mean value and the root-mean-square value of a function
3. Volumes of revolution
4. Lengths of curves and surfaces of revolution

Learning **outcomes**

In this workbook you will learn to interpret an integral as the limit of a sum. You will learn how to apply this approach to the meaning of an integral to calculate important attributes of a curve: the area under the curve, the length of a curve segment, the volume and surface area obtained when a segment of a curve is rotated about an axis. Other quantities of interest which can also be calculated using integration is the position of the centre of mass of a plane lamina and the moment of inertia of a lamina about an axis. You will also learn how to determine the mean value of an integral.

Time **allocation**

You are expected to spend approximately ten hours of independent study on the material presented in this workbook. However, depending upon your ability to concentrate and on your previous experience with certain mathematical topics this time may vary considerably.

Integration as the limit of a sum

14.1



Introduction

In Workbook 13, integration was introduced as the reverse of differentiation. A more rigorous treatment would show that integration is a process of adding or ‘summation’. By viewing integration from this perspective it is possible to apply the techniques of integration to finding areas, volumes, centres of gravity and many other important quantities.

The content of this section is important because it is here that integration is defined rigorously. A thorough understanding of the process involved is essential if you need to apply integration techniques to practical problems.



Prerequisites

① be able to calculate definite integrals

Before starting this Section you should ...



Learning Outcomes

After completing this Section you should be able to ...

- ✓ explain integration as the limit of a sum
- ✓ evaluate the limit of a sum in simple cases

1. The limit of a sum

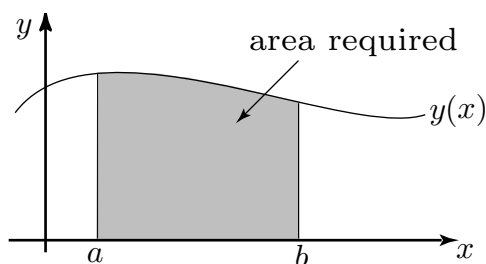


Figure 1. The area under a curve

Consider the graph of the positive function $y(x)$ shown in Figure 1. Suppose we are interested in finding the area under the graph between $x = a$ and $x = b$. One way in which this area can be approximated is to divide it into a number of rectangles, find the area of each rectangle, and then add up all these individual rectangular areas. This is illustrated in Figure 2a, which shows the area divided into n rectangles, and Figure 2b which shows the dimensions of a typical rectangle which is located at $x = x_k$.

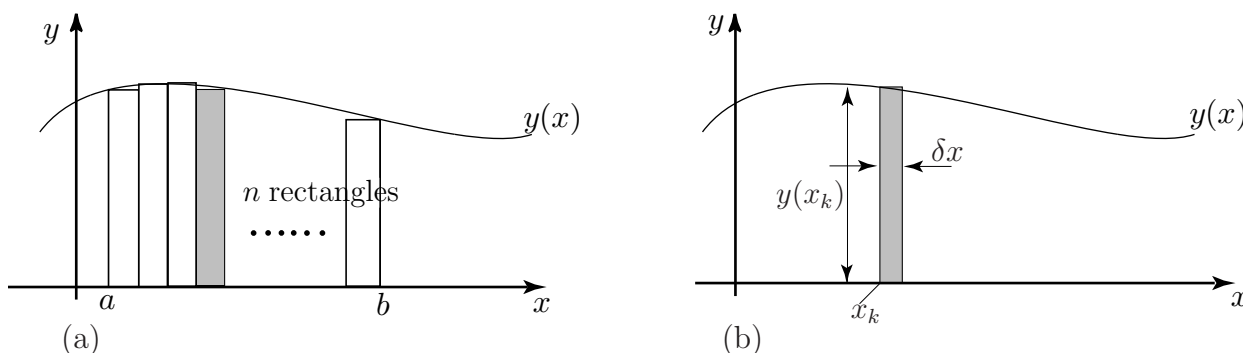


Figure 2. a) The area approximated by n rectangles. b) A typical rectangle.



Firstly, refer to Figure 2a and state the distance from a to b :

Your solution

$$b - a$$

In Figure 2a) the area has been divided into n rectangles. If n rectangles span the distance from a to b state the width of each rectangle:

Your solution

$$\frac{b-a}{n}$$

It is conventional to label the width of each rectangle as δx , i.e. $\delta x = \frac{b-a}{n}$. Suppose we label the x coordinates at the left hand side of the rectangles as x_1, x_2 up to x_n (here $x_1 = a$ and $x_{n+1} = b$). A typical rectangle, the k th rectangle, is shown in Figure 2b). Note that its height is $y(x_k)$. Calculate its area:

Your solution

$x_0 \times (x)h$

The sum of the areas of all n rectangles is then

$$y(x_1)\delta x + y(x_2)\delta x + y(x_3)\delta x + \cdots + y(x_n)\delta x$$

which we write concisely using sigma notation as

$$\sum_{k=1}^n y(x_k)\delta x$$

This quantity gives us an estimate of the area under the curve but it is not exact. To improve the estimate we must take a large number of very thin rectangles. So, what we want to find is the value of this sum when n tends to infinity and δx tends to zero. We write this value as

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n y(x_k)\delta x$$

The lower and upper limits on the sum correspond to the first and last rectangle where $x = a$ and $x = b$ respectively and so we can write this limit in the equivalent form

$$\lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} y(x)\delta x \quad (1)$$

Here, as the number of rectangles increase without bound we drop the subscript k from x_k and write $y(x)$ which is the value of y at a ‘typical’ value of x . If this sum can actually be found, it is called the **definite integral** of $y(x)$, from $x = a$ to $x = b$ and it is written $\int_a^b y(x)dx$. You are already familiar with the technique for evaluating definite integrals which was studied in Workbook 14, section 2.

Therefore we have the following definition:



Key Point

The definite integral $\int_a^b y(x)dx$ is defined as $\lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} y(x)\delta x$

Note that the quantity δx represents the thickness of a small but finite rectangle. When we have taken the limit as δx tends to zero to obtain the integral, we write dx .

This process of dividing an area into very small regions, performing a calculation on each region, and then adding the results by means of an integral is very important. This will become apparent when finding volumes, centres of gravity, moments of inertia etc in the following sections where similar procedures are followed.

Example The area under the graph of $y = x^2$ between $x = 0$ and $x = 1$ is to be found using the technique just described. If the required area is approximated by a large number of thin rectangles, the limit of the sum of their areas is given from Equation 1 as

$$\lim_{\delta x \rightarrow 0} \sum_{x=0}^{x=1} y(x) \delta x$$

Write down the integral which this sum defines and evaluate it to obtain the area under the curve.

Solution

The limit of the sum defines the integral $\int_0^1 y(x)dx$. Here $y = x^2$ and so

$$\int_0^1 x^2 dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3}$$

To show that the process of taking the limit of a sum actually works we discuss the problem in detail.

Example Use the idea of the limit of a sum to find the area under the graph of $y = x^2$ between $x = 0$ and $x = 1$.

Solution

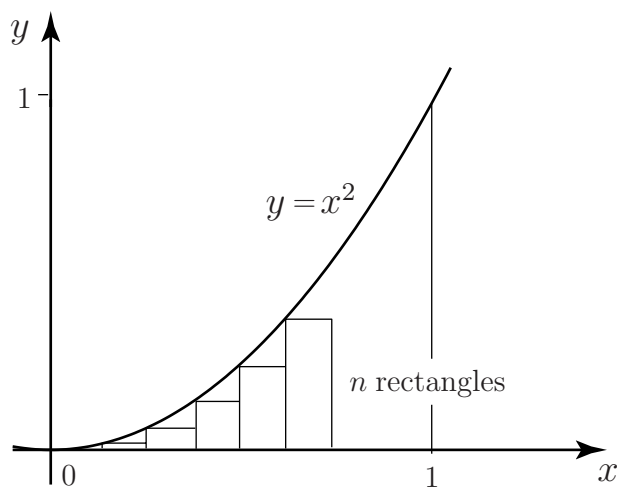


Figure 3. The area under $y = x^2$ is approximated by a number of thin rectangles.



Refer to the graph in Figure 3 to help you answer the questions.

If the interval between $x = 0$ and $x = 1$ is divided into n rectangles what is the width of each rectangle ?

Your solution

$\frac{1}{n}$

Mark this on Figure 3.

What is the x coordinate at the left-hand side of the first rectangle ?

Your solution

0

What is the x coordinate at the left-hand side of the second rectangle ?

Your solution

$\frac{1}{n}$

What is the x coordinate at the left-hand side of the third rectangle ?

Your solution

$$\frac{u}{z}$$

Mark these coordinates on Figure 3.

What is the x coordinate at the left-hand side of the k th rectangle ?

Your solution

$$\frac{u}{1-y}$$

Given that $y = x^2$, what is the y coordinate at the left-hand side of the k th rectangle ?

Your solution

$$z \left(\frac{u}{1-y} \right)$$

The area of the k th rectangle is its height \times its width. Write down the area of the k th rectangle:

Your solution

$$\frac{z^2 u}{z(1-y)} = \frac{u}{1} \times z \left(\frac{u}{1-y} \right)$$

To find the total area A_n of the n rectangles we must add up all these individual rectangular areas:

$$A_n = \sum_{k=1}^n \frac{(k-1)^2}{n^3}$$

This sum can be simplified and then calculated as follows. You will need to make use of the formulas for the sum of the first k integers, and the sum of the squares of the first k integers:

$$\sum_{k=1}^n 1 = n, \quad \sum_{k=1}^n k = \frac{1}{2}n(n+1), \quad \sum_{k=1}^n k^2 = \frac{1}{6}n(n+1)(2n+1)$$

Then, the total area of the rectangles is given by

$$\begin{aligned}
 A_n &= \sum_{k=1}^n \frac{(k-1)^2}{n^3} = \frac{1}{n^3} \sum_{k=1}^n (k-1)^2 \\
 &= \frac{1}{n^3} \sum_{k=1}^n (k^2 - 2k + 1) \\
 &= \frac{1}{n^3} \left(\sum_{k=1}^n k^2 - 2 \sum_{k=1}^n k + \sum_{k=1}^n 1 \right) \\
 &= \frac{1}{n^3} \left(\frac{n}{6}(n+1)(2n+1) - 2 \frac{n}{2}(n+1) + n \right) \\
 &= \frac{1}{n^2} \left(\frac{(n+1)(2n+1)}{6} - (n+1) + 1 \right) \\
 &= \frac{1}{n^2} \left(\frac{(n+1)(2n+1)}{6} - n \right) \\
 &= \frac{1}{n^2} \left(\frac{(n+1)(2n+1)}{6} - \frac{6n}{6} \right) \\
 &= \frac{1}{6n^2} (2n^2 - 3n + 1) \\
 &= \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}
 \end{aligned}$$

Note that this is a formula for the total area of the n rectangles. It is an estimate of the area under the graph of $y = x^2$. Now, as n gets larger, the terms $\frac{1}{2n}$ and $\frac{1}{6n^2}$ become small and will eventually tend to zero. If we now let n tend to infinity we obtain the exact answer:

Your solution

$$\lim_{n \rightarrow \infty} \left(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2} \right) =$$

$\frac{1}{3}$

The required area is $\frac{1}{3}$.

The area has been found as **the limit of a sum** and of course agrees with that calculated by integration.

In the calculations which follow in subsequent sections the need to evaluate complicated limits like this is avoided by performing the integration using the techniques of Workbook 14. Nevertheless it will still be necessary to go through the process of dividing a region into small sections, performing a calculation on each section and then adding the results, in order to formulate the integral required.

Exercises

There are deliberately few exercises in this section because in practice integrals are evaluated using the techniques of Workbook 14 and not by taking explicit limits of sums. What **is** important though is an understanding of how the appropriate sum is formed.

1. Find the area under $y = x + 1$ from $x = 0$ to $x = 10$ using the limit of a sum.
2. Find the area under $y = 3x^2$ from $x = 0$ to $x = 2$ using the limit of a sum.
3. Write down, but do not evaluate, the integral defined by the limit as $\delta x \rightarrow 0$, or $\delta t \rightarrow 0$ of the following sums:

$$(a) \sum_{x=0}^{x=1} x^3 \delta x, \quad (b) \sum_{x=0}^{x=4} 4\pi x^2 \delta x, \quad (c) \sum_{t=0}^{t=1} t^3 \delta t, \quad (d) \sum_{x=0}^{x=1} 6mx^2 \delta x.$$

Answers 1. 60. 2. 8. 3(a) $\int_1^0 x^3 dx$. (b) $\int_4^0 4\pi x^2 dx$. (c) $\int_1^0 t^3 dt$, (d) $\int_1^0 6mx^2 dx$.