The mean value and the root-mean-square value of a function

14.2

Introduction

Currents and voltages often vary with time and engineers may wish to know the average value of such a current or voltage over some particular time interval. The average value of a time-varying function is defined in terms of an integral. An associated quantity is the **root-mean-square** (r.m.s) value of a current which is used, for example, in the calculation of the power dissipated by a resistor.

Prerequisites

Before starting this Section you should ...

① be able to calculate definite integrals

② be familiar with a table of trigonometric identities

Learning Outcomes

After completing this Section you should be able to ...

✓ calculate the mean value of a function

✓ calculate the root-mean-square value of a function

1. Average value of a function

Suppose a time-varying function $f(t)$ is defined on the interval $a \le t \le b$. The area, A, under the graph of $f(t)$ is given by the integral

$$A = \int_a^b f(t)\mathrm{d}t$$

This is illustrated in Figure 1.

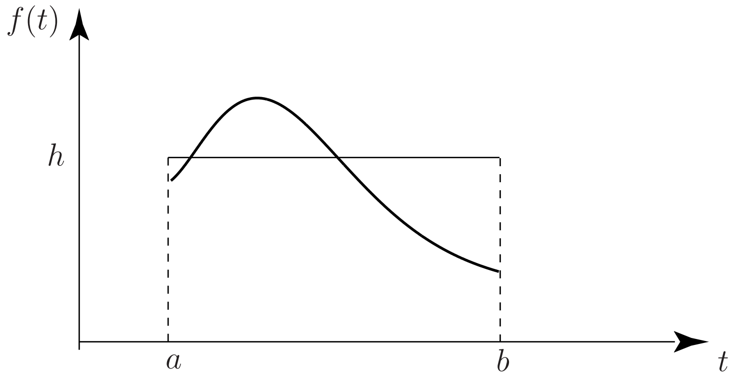

Figure 1. the area under the curve from $t = a$ to $t = b$ and the area of the rectangle are equal

On Figure 1 we have also drawn a rectangle with base spanning the interval $a \le t \le b$ and which has the same area as that under the curve. Suppose the height of the rectangle is h. Then

$$\begin{aligned} \text{area of rectangle} &= \text{area under curve} \\ h(b-a) &= \int_a^b f(t)\mathrm{d}t \\ h &= \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \end{aligned}$$

The value of h is the **average or mean value** of the function across the interval $a \le t \le b$.

Key Point

The **average value** of a function $f(t)$ in the interval $a \le t \le b$ is

$$\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t$$

The average value depends upon the interval chosen. If the values of a or b are changed, then the average value of the function across the interval from a to b will change as well.

Example Find the average value of $f(t) = t^2$ over the interval $1 \leq t \leq 3$.

Solution

Here $a = 1$ and $b = 3$.

$$\begin{aligned}\text{average value} &= \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \\ &= \frac{1}{3-1}\int_1^3 t^2\mathrm{d}t = \frac{1}{2}\left[\frac{t^3}{3}\right]_1^3 = \frac{13}{3}\end{aligned}$$

Find the average value of $f(t) = t^2$ over the interval $2 \leq t \leq 5$.

Here $a = 2$ and $b = 5$.

Your solution

average value =

$\frac{1}{5-2}\int_2^5 t^2\mathrm{d}t$

Now evaluate the integral.

Your solution

average value =

13

Exercises

1. Calculate the average value of the given functions across the specified interval.
   (a) $f(t) = 1 + t$ across $[0, 2]$
   (b) $f(x) = 2x - 1$ across $[-1, 1]$
   (c) $f(t) = t^2$ across $[0, 1]$
   (d) $f(t) = t^2$ across $[0, 2]$
   (e) $f(z) = z^2 + z$ across $[1, 3]$

2. Calculate the average value of the given functions over the specified interval.
   (a) $f(x) = x^3$ across $[1, 3]$
   (b) $f(x) = \frac{1}{x}$ across $[1, 2]$
   (c) $f(t) = \sqrt{t}$ across $[0, 2]$
   (d) $f(z) = z^3 - 1$ across $[-1, 1]$
   (e) $f(t) = \frac{1}{t^2}$ across $[-3, -2]$

3. Calculate the average value of the following:
   (a) $f(t) = \sin t$ across $\left[0, \frac{\pi}{2}\right]$
   (b) $f(t) = \sin t$ across $[0, \pi]$
   (c) $f(t) = \sin \omega t$ across $[0, \pi]$
   (d) $f(t) = \cos t$ across $\left[0, \frac{\pi}{2}\right]$
   (e) $f(t) = \cos t$ across $[0, \pi]$
   (f) $f(t) = \cos \omega t$ across $[0, \pi]$
   (g) $f(t) = \sin \omega t + \cos \omega t$ across $[0, 1]$

4. Calculate the average value of the following functions:
   (a) $f(t) = \sqrt{t+1}$ across $[0, 3]$
   (b) $f(t) = e^t$ across $[-1, 1]$
   (c) $f(t) = 1 + e^t$ across $[-1, 1]$

Answers 1. (a) 2 (b) -1 (c) $\dfrac{1}{3}$ (d) $\dfrac{4}{3}$ (e) $\dfrac{19}{3}$

2. (a) 10 (b) 0.6931 (c) 0.9428 (d) -1 (e) $\dfrac{1}{6}$

3. (a) $\dfrac{2}{\pi}$ (b) $\dfrac{2}{\pi}$ (c) $\dfrac{1}{\pi\omega}[1 - \cos(\pi\omega)]$ (d) $\dfrac{2}{\pi}$ (e) 0 (f) $\dfrac{\sin(\pi\omega)}{\pi\omega}$

(g) $\dfrac{1 + \sin\omega - \cos\omega}{\omega}$

4. (a) $\dfrac{14}{9}$ (b) 1.1752 (c) 2.1752

2. Root-mean-square value of a function.

If $f(t)$ is defined on the interval $a \le t \le b$, the **mean-square value** is given by the expression:

$$\frac{1}{b-a}\int_a^b [f(t)]^2 \mathrm{d}t$$

This is simply the average value of $[f(t)]^2$ over the given interval.

The related quantity: the **root-mean-square** (r.m.s.) value is given by the following formula.

Key Point

$$\text{r.m.s value } = \sqrt{\frac{1}{b-a}\int_a^b [f(t)]^2 \mathrm{d}t}$$

The r.m.s. value depends upon the interval chosen. If the values of a or b are changed, then the r.m.s value of the function across the interval from a to b will change as well. Note that when finding an r.m.s. value the function must be squared before it is integrated.

Example Find the r.m.s. value of $f(t) = t^2$ across the interval from $t = 1$ to $t = 3$.

Solution

$$\begin{aligned}
\text{r.m.s} &= \sqrt{\frac{1}{b-a}\int_a^b [f(t)]^2 \mathrm{d}t} \\
&= \sqrt{\frac{1}{3-1}\int_1^3 [t^2]^2 \mathrm{d}t} \\
&= \sqrt{\frac{1}{2}\int_1^3 t^4 \mathrm{d}t} = \sqrt{\frac{1}{2}\left[\frac{t^5}{5}\right]_1^3} = 4.92
\end{aligned}$$

Example Calculate the r.m.s value of $f(t) = \sin t$ across the interval $0 \le t \le 2\pi$.

Solution

Here $a = 0$ and $b = 2\pi$.

$$\text{r.m.s} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} \sin^2 t \,\mathrm{d}t}$$

The integral of $\sin^2 t$ is performed by using trigonometrical identities to rewrite it in the alternative form $\frac{1}{2}(1 - \cos 2t)$. This technique was described in Chapter 13 section 7.

$$\begin{aligned} \text{r.m.s. value} &= \sqrt{\frac{1}{2\pi}\int_0^{2\pi} \frac{(1-\cos 2t)}{2}\mathrm{d}t} \\ &= \sqrt{\frac{1}{4\pi}\int_0^{2\pi} (1-\cos 2t)\mathrm{d}t} \\ &= \sqrt{\frac{1}{4\pi}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi}} \\ &= \sqrt{\frac{1}{4\pi}(2\pi)} = \sqrt{\frac{1}{2}} = 0.707 \end{aligned}$$

Thus the r.m.s value is 0.707.

In the previous example the amplitude of the sine wave was 1, and the r.m.s. value was 0.707. In general, if the amplitude of a sine wave is A, its r.m.s value is $0.707A$.

Key Point

The r.m.s value of any sinusoidal waveform taken across an interval equal to one period is $0.707 \times$ amplitude of the waveform.

Exercises

1. Calculate the r.m.s. values of the functions in question 1 of the previous Exercises.
2. Calculate the r.m.s. values of the functions in question 2 of the previous Exercises.
3. Calculate the r.m.s. values of the functions in question 3 in the previous Exercises.
4. Calculate the r.m.s. values of the functions in question 4 in the previous Exercises.

Answers 1. (a) 2.0817 (b) 1.5275 (c) 0.4472 (d) 1.7889 (e) 6.9666

2. (a) 12.4957 (b) 0.7071 (c) 1 (d) 1.0690 (e) 0.1712

3. (a) 0.7071 (b) 0.7071 (c) $\sqrt{\frac{1}{2} - \frac{\sin \pi\omega \cos \pi\omega}{2\pi\omega}}$ (d) 0.7071 (e) 0.7071 (f) $\sqrt{\frac{1}{2} + \frac{\sin \pi\omega \cos \pi\omega}{2\pi\omega}}$

(g) $\sqrt{1 + \frac{\sin^2 \omega}{\omega}}$

4. (a) 1.5811 (b) 1.3466 (c) 2.2724