Volumes of revolution 14.3

Introduction

In this section we show how the concept of integration as the limit of a sum can be used to find volumes of solids formed when curves are rotated around the x or y axes.

Prerequisites

Before starting this Section you should . . .

① be able to calculate definite integrals

② understand integration as the limit of a sum

Learning Outcomes

After completing this Section you should be able to . . .

✓ calculate volumes of revolution

1. Volumes generated by rotating curves about the x-axis

Figure 1 shows a graph of the function $y = 2x$ for x between 0 and 3.

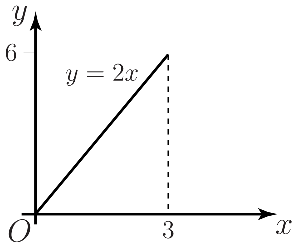

Figure 1. A graph of the function $y = 2x$, for $0 \le x \le 3$.

Imagine rotating the line $y = 2x$ by one complete revolution (360^0 or 2π radians) around the x-axis. The surface so formed is the surface of a cone as shown in Figure 2. Such a three-dimensional shape is known as a **solid of revolution**. We now discuss how to obtain the volumes of such solids of revolution.

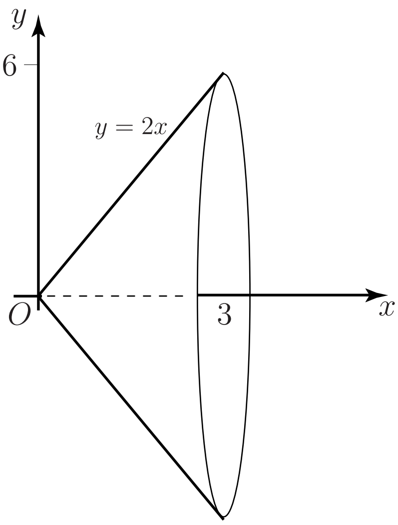

Figure 2. When the line $y = 2x$ is rotated around the axis, a solid is generated.

Find the volume of the cone generated by rotating $y = 2x$, for $0 \le x \le 3$, around the x-axis, as shown in Figure 2.

In order to find the volume of this solid we assume that it is composed of lots of thin circular discs all aligned perpendicular to the x-axis, such as that shown in Figure 3. From Figure 3 we note that a typical disc has radius y, which in this example equals $2x$, and thickness δx.

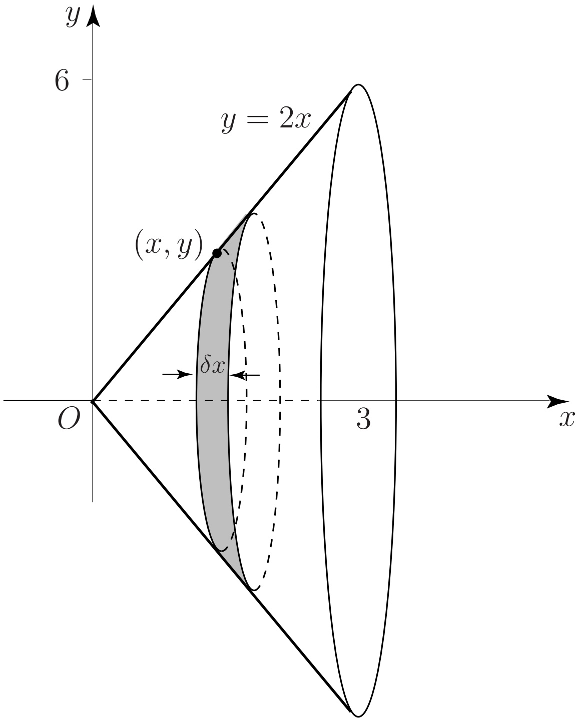

Figure 3. The cone is divided into a number of thin circular discs.

The volume of a circular disc is the circular area multiplied by the thickness.

Write down an expression for the volume of this typical disc:

Your solution

$\pi(2x)^2\delta x = 4\pi x^2\delta x$

To find the total volume we must sum the contributions from all discs and find the limit of this sum as the number of discs becomes infinite and δx becomes zero. That is

$$\lim_{\delta x \to 0} \sum_{x=0}^{x=3} 4\pi x^2 \delta x$$

This is the definition of a definite integral. Write down the corresponding integral.

Your solution

$\int_0^3 4\pi x^2 \, dx$

Find the required volume by performing the integration:

Your solution

$\left[\frac{4\pi x^3}{3}\right]_0^3 = 36\pi$

Now let us do another example.

A graph of the function $y = x^2$ for x between 0 and 4 is shown in Figure 4. The graph is rotated around the x-axis to produce the solid shown. Find its volume.

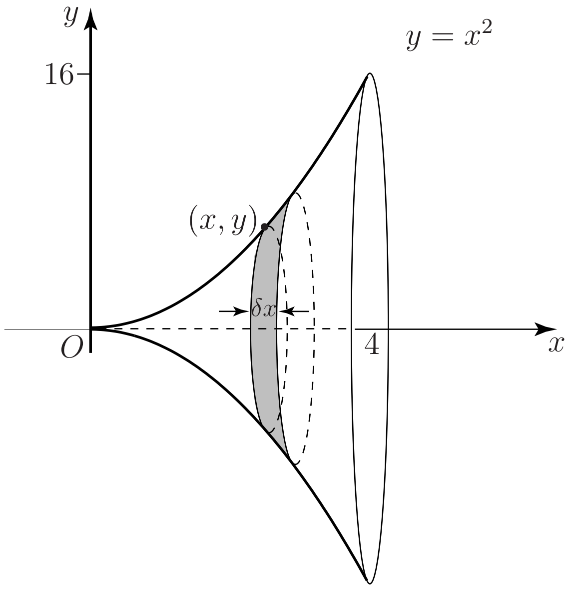

Figure 4. The solid of revolution is divided into a number of thin circular discs.

As in the previous guided exercise, the solid is considered to be composed of lots of circular discs of radius y, (which in this example is equal to x^2), and thickness δx.

Write down the volume of each disc:

Your solution

$\pi(x^2)^2\,\delta x = \pi x^4\,\delta x$

Write down the expression which results by summing the volumes of all such discs:

Your solution

$\sum_{x=0}^{x=4} \pi x^4\,\delta x$

Write down the integral which results from taking the limit of the sum as $\delta x \to 0$:

Your solution

$\int_0^4 \pi x^4\,dx$

Perform the integration to find the volume of the solid:

Your solution

$\frac{4^5\pi}{5} = 204.8\pi$

In general, suppose the graph of $y(x)$ between $x = a$ and $x = b$ is rotated about the x-axis, and the solid so formed is considered to be composed of lots of circular discs of thickness δx.

Write down an expression for the radius of a typical disc:

Your solution

y

Write down an expression for the volume of a typical disc:

Your solution

$\pi y^2 \delta x$

The total volume is found by summing these individual volumes and taking the limit as δx tends to zero:

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 \delta x$$

Write down the definite integral which this sum defines:

Your solution

$\int_a^b \pi y^2 \mathrm{d}x$

Key Point

If the graph of $y(x)$, between $x = a$ and $x = b$, is rotated about the x-axis the volume of the solid formed is

$$\int_a^b \pi y^2 \mathrm{d}x$$

Exercises

1. When the graph of $y(x)$ between $x = a$ and $x = b$ is rotated around the x-axis, show that the volume of the solid formed is $\int_a^b \pi\, y^2\, dx$.

2. Find the volume of the solid formed when that part of the curve between $y = x^2$ between $x = 1$ and $x = 2$ is rotated about the x-axis.

3. The parabola $y^2 = 4x$ for $0 \leq x \leq 1$, is rotated around the x-axis. Find the volume of the solid formed.

Answers 2. $31\pi/5$, 3. 2π.

2. Volumes generated by rotating curves about the y-axis

We can obtain a different solid of revolution by rotating a curve around the y-axis instead of around the x-axis. See Figure 5.

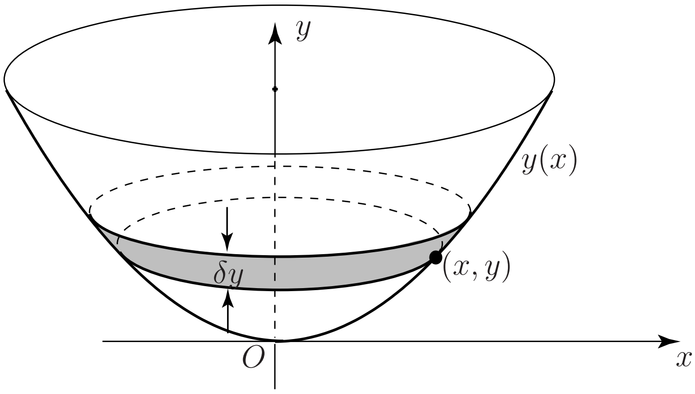

Figure 5. A solid can be generated by rotation around the y-axis.

To find the volume of this solid it is divided into a number of circular discs as before, but this time the discs are horizontal. The radius of a typical disc is x and its thickness is δy. The volume of the disc will be $\pi x^2 \delta y$ where δy is the thickness of the disc.
The total volume is found by summing these individual volumes and taking the limit as $\delta y \to 0$. If the lower and upper limits on y are c and d, we obtain for the volume:

$$\lim_{\delta y \to 0} \sum_{y=c}^{y=d} \pi x^2 \delta y$$

which is the definite integral

$$\int_c^d \pi x^2 \mathrm{d}y$$

Key Point

If the graph of $y(x)$, between $y = c$ and $y = d$, is rotated about the y-axis the volume of the solid formed is

$$\int_c^d \pi x^2 \mathrm{d}y$$

Find the volume generated when the graph of $y = x^2$ between $x = 0$ and $x = 1$ is rotated around the y-axis.

On the graph of $y = x^2$, when $x = 0$, $y = 0$ and when $x = 1$ $y = 1$ and so the limits on y are the same as the limits on x.

Write down the required integral.

Your solution

$\int_0^1 \pi x^2 \mathrm{d}y$

Because $y = x^2$ this integral can be written entirely in terms of y. Do this now, and then evaluate the integral.

Your solution

$\int_0^1 \pi y \mathrm{d}y = \left[\frac{\pi y^2}{2}\right]_0^1 = \frac{\pi}{2}$

Exercises

1. When the graph of $y(x)$ between $x = a$ and $x = b$ is rotated around the y-axis, show that the volume of the solid formed is $\int_c^d \pi x^2 \,\mathrm{d}y$ where $c = y(a)$ and $d = y(b)$.
2. The curve $y = x^2$ for $1 < x < 2$ is rotated about the y-axis. Find the volume of the solid formed.
3. The line $y = 2 - 2x$ for $0 \leq x \leq 2$ is rotated around the y-axis. Find the volume of revolution.

Answers 2. $\frac{15\pi}{2}$. 3. $\frac{16\pi}{3}$.