Lengths of curves and surfaces of revolution

14.4

Introduction

Integration can be used to find the length of a curve and the area of the surface generated when a curve is rotated around an axis. In this section we state and use the formulae for doing this.

Prerequisites

Before starting this Section you should . . .

① be able to calculate definite integrals

Learning Outcomes

After completing this Section you should be able to . . .

✓ find the length of a number of curves

✓ find the area of the surface generated when a curve is rotated about an axis

1. The length of a curve

To find the length of a curve in the xy plane we first divide the curve into a large number of pieces. We measure (or, at least, approximate) the length of each piece and then by an obvious summation process obtain an estimate for the length of the curve. Theoretically, we allow the number of pieces to increase without bound, implying that the length of each piece will tend to zero. In this limit the summation process becomes an integration process.

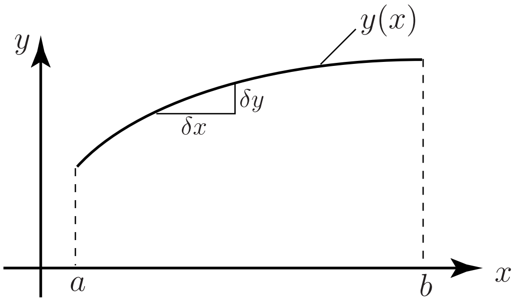

Figure 1

Figure 1 shows the portion of the curve $y(x)$ between $x = a$ and $x = b$. A small piece of this curve has been selected and can be considered as the hypotenuse of a triangle with base δx and height δy. (Here δx and consequently, δy are intended to be 'small' so that the *curved* piece can be regarded as a *straight* piece).

Using Pythagoras' theorem, the length of the hypotenuse is:

$$\sqrt{\delta x^2 + \delta y^2} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x$$

By summing all such contributions between $x = a$ and $x = b$, and letting $\delta x \to 0$ we obtain an expression for the total length of the curve:

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x$$

But we already know how to write such an expression in terms of an integral. We obtain the following result:

Key Point

Given a curve with equation $y = f(x)$, then the length of the curve between the points where $x = a$ and $x = b$ is given by the formula:

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,\mathrm{d}x$$

Because of the form of the integrand, and in particular the square root, integrals of this type are often difficult to calculate and in practice, approximate rather than exact methods are normally needed to perform the integration. We shall illustrate the application of the formula by an example which could be calculated in a much simpler way, before looking at some harder examples.

Find the length of the curve $y = 3x + 2$ between $x = 1$ and $x = 5$.

In this example, the curve is in fact a straight line segment, and its length could be obtained using methods other than integration.

Notice from the formula in the Key Point that it is necessary to find $\frac{dy}{dx}$. Do this first.

Your solution

$\frac{dy}{dx} = 3$

Applying the formula we find

$$\begin{aligned} \text{length of curve} &= \int_1^5 \sqrt{1 + (3)^2}\,\mathrm{d}x \\ &= \int_1^5 \sqrt{10}\,\mathrm{d}x \\ &= \left[\sqrt{10}x\right]_1^5 \\ &= (5-1)\sqrt{10} = 4\sqrt{10} = 12.65 \end{aligned}$$

Thus the length of the curve $y = 3x + 2$ between the points where $x = 1$ and $x = 5$ is 12.65 units.

Find the length of the curve $y = \cosh x$ between $x = 0$ and $x = 2$ shown in Figure 2.

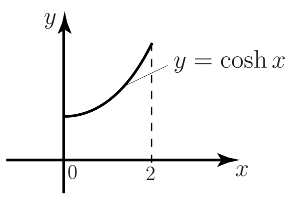

Figure 2

First write down $\frac{dy}{dx}$

Your solution

$\frac{dy}{dx} =$

$\frac{dy}{dx} = \sinh x$

Hence write down the required integral:

Your solution

$\int_0^2 \sqrt{1 + \sinh^2 x}\, dx$

This integral can be evaluated by making use of the hyperbolic identity

$$\cosh^2 x - \sinh^2 x = 1$$

Write down the integral which results after applying this identity.

Your solution

$\int_0^2 \cosh x dx$

Perform the integration for yourself to find the required length.

Your solution

$[\sinh x]_0^2 = 3.63$

Thus the length of $y = \cosh x$ between $x = 0$ and $x = 2$ is 3.63 units.

The final example is more complicated still and requires the use of a hyperbolic substitution and knowledge of the hyperbolic identities.

Find the length of the curve $y = x^2$ between $x = 0$ and $x = 3$.
Given $y = x^2$ then $\frac{dy}{dx} = 2x$. Apply the formula to obtain the integral required:

Your solution

$\int_0^3 \sqrt{1 + 4x^2} dx$

Make the substitution $x = \frac{1}{2}\sinh u$, $\frac{dx}{du} = \frac{1}{2}\cosh u$, to obtain an integral in terms of u.

Your solution

$\int_0^{\sinh^{-1}6} \sqrt{1+\sinh^2 u}\,\frac{1}{2}\cosh u\,du$

Use the hyperbolic identity $\cosh^2 u - \sinh^2 u = 1$ to rewrite this integral:

Your solution

$\frac{1}{2}\int_0^{\sinh^{-1}6} \cosh^2 u\,du$

Another hyperbolic identity is $\cosh^2 u = \frac{1}{2}(\cosh 2u + 1)$. Apply this identity to rewrite the integrand.

Your solution

$\frac{1}{4}\int_0^{\sinh^{-1}6} (\cosh 2u + 1)\,du$

Finally, performing the integration we can now complete the calculation:

$$\begin{aligned} \frac{1}{4}\int_0^{\sinh^{-1}6} (\cosh 2u + 1)\,du &= \frac{1}{4}\left[\frac{\sinh 2u}{2} + u\right]_0^{\sinh^{-1}6} \\ &= 9.75 \end{aligned}$$

Thus the length of the curve $y = x^2$ between $x = 0$ and $x = 3$ is 9.75 units.

Exercises

1. Find the length of the line $y = 2x + 7$ between $x = 1$ and $x = 3$ using the technique of this section. Verify your result from your knowledge of the straight line.
2. Find the length of $y = x^{3/2}$ between $x = 0$ and $x = 5$.
3. Calculate the length of the curve $y^2 = 4x^3$ between $x = 0$ and $x = 2$.

Answers 1. $2\sqrt{5} = 4.47$ 2. 12.41 3. 6.06 (first quadrant only).

2. The area of a surface of revolution.

In section 2 we found an expression for the volume of a solid of revolution. Here we consider the more complicated problem of formulating an expression for the surface area of a solid of revolution.

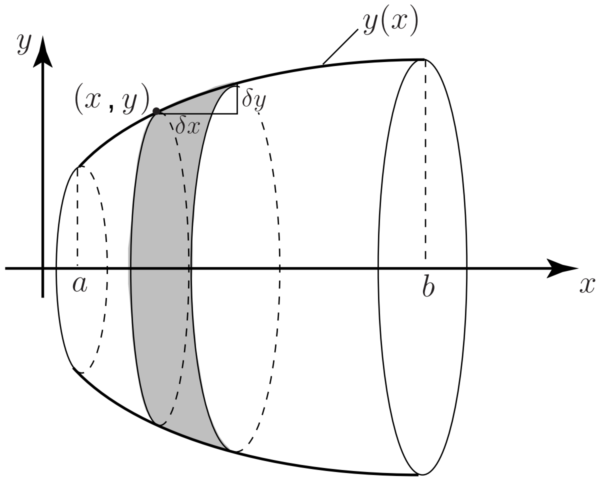

Figure 3

Figure 3 shows the portion of the curve $y(x)$ between $x = a$ and $x = b$ which is rotated around the x axis through 360°. A small disc, of thickness δx, of the solid of revolution has been selected. Its radius is y and so its circumference has length $2\pi y$. (As usual we assume δx is 'small' so that the *curved* part of $y(x)$ representing the hypotenuse of the highlighted 'triangle' can be regarded as *straight*). This surface 'ribbon', shown shaded, has a length $2\pi y$ and a width $\sqrt{(\delta x)^2 + (\delta y)^2}$ and so its area is then, to a good approximation, $2\pi y\sqrt{(\delta x)^2 + (\delta y)^2}$. We now let $\delta x \to 0$ to obtain the result:

Key Point

Given a curve with equation $y = f(x)$, then the surface area of the solid generated by rotating that part of the curve between the points where $x = a$ and $x = b$ around the x axis is given by the formula:

$$\text{area of surface } = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,\mathrm{d}x$$

Find the area of the surface generated when the part of the curve $y = x^3$ between $x = 0$ and $x = 4$ is rotated around the x axis.

The area of surface is given by

$$
\begin{aligned}
\text{area} &= \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,\mathrm{d}x \\
&= \int_0^4 2\pi x^3 \sqrt{1+(3x^2)^2}\,\mathrm{d}x = \int_0^4 2\pi x^3\sqrt{1+9x^4}\,\mathrm{d}x
\end{aligned}
$$

This integral can be found by making a substitution $u = 1 + 9x^4$, $\frac{du}{dx} = 36x^3$ so that $x^3\mathrm{d}x = \frac{du}{36}$. When $x = 0$, $u = 1$ and when $x = 4$, $u = 2305$.

Write down the corresponding integral in terms of u.

Your solution

$\frac{\pi}{18}\int_1^{2305}\sqrt{u}\,du$

Perform the integration.

Your solution

$\frac{\pi}{18}\left[\frac{2}{3}\right]_1^{2305}$

Apply the limits of integration to find the area.

Your solution

$\frac{\pi}{27}\left((2305)^{3/2}-1\right)$

Exercises

1. The line $y = x$ between $x = 0$ and $x = 1$ is rotated around the x axis.
   (a) Find the area of the surface generated.
   (b) Verify this result by finding the curved surface area of the corresponding cone. (The curved surface area of a cone of radius r and slant height ℓ is $\pi r\ell$.)
2. Find the area of the surface generated when $y = \sqrt{x}$ for $1 \le x \le 2$ is rotated completely about the x axis.

Answers 1. $\pi\sqrt{2}$ 2. 8.28.