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applications of integration 2

Learning outcomes

In this workbook you will learn to interpret an integral as the limit of a sum. You will learn how to apply this approach to the meaning of an integral to calculate important attributes of a curve: the area under the curve, the length of a curve segment, the volume and surface area obtained when a segment of a curve is rotated about an axis. Other quantities of interest which can also be calculated using integration is the position of the centre of mass of a plane lamina and the moment of inertia of a lamina about an axis. You will also learn how to determine the mean value of an integal.

Time allocation

You are expected to spend approximately ten hours of independent study on the material presented in this workbook. However, depending upon your ability to concentrate and on your previous experience with certain mathematical topics this time may vary considerably.

Integration of Vectors

15.1

Introduction

The area known as vector calculus is used to model mathematically a vast range of engineering phenomena including electrostatics, electromagnetic fields, air flow around aircraft and heat flow in nuclear reactors. In this section we introduce briefly the integral calculus of vectors.

Prerequisites

Before starting this Section you should . . .

① have a knowledge of vectors, in cartesian form

② be able to calculate the scalar and vector products of two vectors

③ be able to integrate scalar functions.

Learning Outcomes

After completing this Section you should be able to . . .

✓ integrate vectors

1. Integration of Vectors

If a vector depends upon time t, it is often necessary to integrate it with respect to time. Recall that $\underline{i}$, $\underline{j}$ and $\underline{k}$ are constant vectors and must be treated thus in any integration. Hence the integral,

$$\underline{I} = \int (f(t)\underline{i} + g(t)\underline{j} + h(t)\underline{k})\,\mathrm{d}t$$

is simply evaluated as three scalar integrals i.e.

$$\underline{I} = \left(\int f(t)\mathrm{d}t\right)\underline{i} + \left(\int g(t)\mathrm{d}t\right)\underline{j} + \left(\int h(t)\mathrm{d}t\right)\underline{k}$$

Example If $\underline{r} = 3t\underline{i} + t^2\underline{j} + (1+2t)\underline{k}$, evaluate $\displaystyle\int_0^1 \underline{r}\mathrm{d}t$.

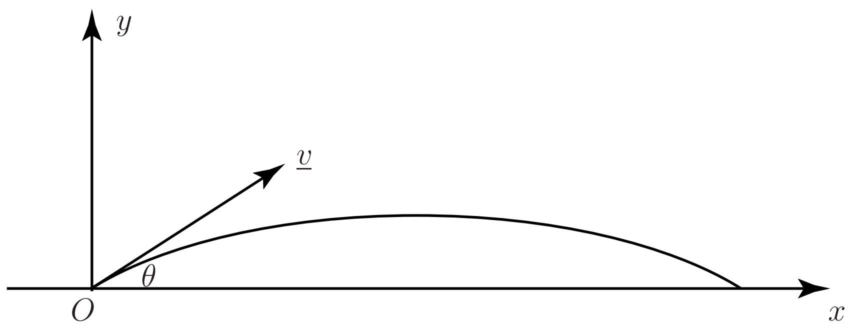
Solution

$$\begin{aligned}\int_0^1 \underline{r}dt &= \left(\int_0^1 3t\,\mathrm{d}t\right)\underline{i} + \left(\int_0^1 t^2\mathrm{d}t\right)\underline{j} + \left(\int_0^1 (1+2t)\,\mathrm{d}t\right)\underline{k} \\ &= \left[\frac{3t^2}{2}\right]_0^1\underline{i} + \left[\frac{t^3}{3}\right]_0^1\underline{j} + \left[t+t^2\right]_0^1\underline{k} = \frac{3}{2}\underline{i} + \frac{1}{3}\underline{j} + 2\underline{k}\end{aligned}$$

Trajectories

To simplify the modelling of the path of a body projected from a fixed point we usually ignore any air resistance and effects due to the wind. Once this initial model is understood other variables and effects can be introduced into the model.

A particle is projected from a point O with velocity $\underline{u}$ and an angle θ above the horizontal as shown in the diagram.



The only force acting on the particle in flight is gravity acting downwards, so if m is the mass of the projectile and taking axes as shown, the force due to gravity is $-mg\underline{j}$. Now using Newton's Second Law (rate of change of momentum is equal to the applied force) we have

$$\frac{\mathrm{d}m\underline{v}}{\mathrm{d}t} = -mg\underline{j}$$

Cancelling the common factor m and integrating we have

$$\underline{v}(t) = -gt\underline{j} + \underline{c} \quad \text{where } \underline{c} \text{ is a } \textit{constant vector}$$

However, velocity is the rate of change of position: $\underline{v}(t) = \dfrac{\mathrm{d}\underline{r}}{\mathrm{d}t}$ so

$$\frac{\mathrm{d}\underline{r}}{\mathrm{d}t} = -gt\underline{j} + \underline{c}$$

Integrating once more:

$$\underline{r}(t) = -\frac{1}{2}gt^2\underline{j} + \underline{c}t + \underline{d} \text{ where } \underline{d} \text{ is another constant vector.}$$

The values of these constant vectors may be determined by using the *initial conditions* in this problem: when $t = 0$ then $\underline{r} = \underline{0}$ and $\underline{v} = \underline{u}$. Imposing these initial conditions gives

$\underline{d} = \underline{0}$ and $\underline{c} = u\cos\theta\underline{i} + u\sin\theta\underline{j}$ where u is the magnitude of $\underline{u}$. This gives

$$\underline{r}(t) = ut\cos\theta\underline{i} + (ut\sin\theta - \frac{1}{2}gt^2)\underline{j}.$$

The interested reader might try to show why the path of the particle is a parabola.

Exercises

1. Given $\underline{r} = 3\sin t\ \underline{i} - \cos t\ \underline{j} + (2-t)\underline{k}$, evaluate $\displaystyle\int_0^{\pi} \underline{r}\,\mathrm{d}t$.

2. Given $\underline{v} = \underline{i} - 3\underline{j} + \underline{k}$, evaluate:

   (a) $\displaystyle\int_0^1 \underline{v}\,\mathrm{d}t$, (b) $\displaystyle\int_0^2 \underline{v}\,\mathrm{d}t$

3. The vector, $\underline{a}$, is defined by $\underline{a} = t^2\underline{i} + e^{-t}\underline{j} + t\underline{k}$. Evaluate

   (a) $\displaystyle\int_0^1 \underline{a}\,\mathrm{d}t$, (b) $\displaystyle\int_2^3 \underline{a}\,\mathrm{d}t$, (c) $\displaystyle\int_1^4 \underline{a}\,\mathrm{d}t$

4. Let $\underline{a}$ and $\underline{b}$ be two three-dimensional vectors. Is the following result true?

$$\int_{t_1}^{t_2} \underline{a}\,\mathrm{d}t \times \int_{t_1}^{t_2} \underline{b}\,\mathrm{d}t = \int_{t_1}^{t_2} \underline{a} \times \underline{b}\,\mathrm{d}t$$

   where $\times$ denotes the vector product.

Answers

1. $6\underline{i} + 1.348\underline{k}$
2. (a) $\underline{i} - 3\underline{j} + \underline{k}$ (b) $2\underline{i} - 6\underline{j} + 2\underline{k}$
3. (a) $0.333\underline{i} + 0.632\underline{j} + 0.5\underline{k}$ (b) $6.333\underline{i} + 0.0855\underline{j} + 2.5\underline{k}$ (c) $21\underline{i} + 0.3496\underline{j} + 7.5\underline{k}$
4. no.