Calculating centres of mass

15.2

Introduction

In this section we show how the idea of integration as the limit of a sum can be used to find the centre of mass of an object such as a thin plate, like a sheet of metal. Such a plate is also known as a *lamina*. An understanding of the term 'moment' is necessary and so this concept is introduced.

Prerequisites

Before starting this Section you should . . .

① understand integration as the limit of a sum

② be able to calculate definite integrals

Learning Outcomes

After completing this Section you should be able to . . .

✓ calculate the position of the centre of mass of a variety of simple plane shapes

1. The centre of mass of a collection of point masses

Suppose we have a collection of masses located at a number of known points along a line. The **centre of mass** is the point where, for many purposes, all the mass can be assumed to be located.

For example, if two objects each of mass m are placed at distances 1 and 2, from a point O as shown in Figure 1a, then the total mass, $2m$, might be assumed to be concentrated at distance 1.5 as shown in Figure 1b. This is the point where we could imagine placing a pivot to achieve a perfectly balanced system.

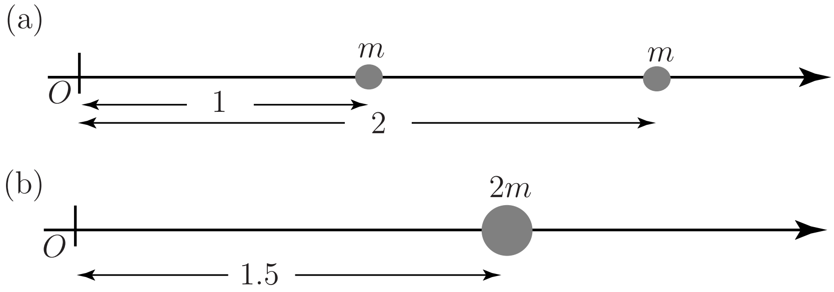

Figure 1. b) shows the location of the centre of mass of the objects in a).

To think of this another way, if a pivot is placed at the origin, as on a see-saw, then the two masses at $x = 1$ and $x = 2$ together have the same **turning effect** or **moment** as a single mass $2m$ located at $x = 1.5$. This is illustrated in Figure 2.

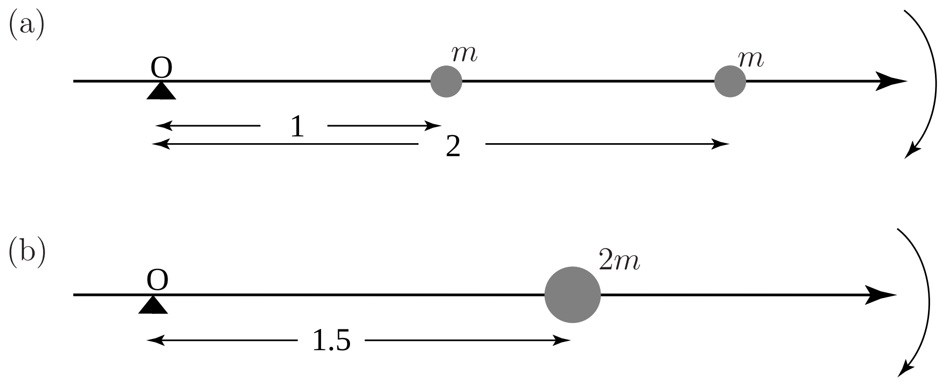

Figure 2. The single object of mass $2m$ has the same turning effect as the two objects each of mass m.

Before we can calculate the position of the centre of mass of a collection of masses it is important to define the term 'moment' more precisely.
Given a mass M located a distance d from O, as shown in Figure 3, its moment about O is defined to be

$$\textbf{moment} = Md$$

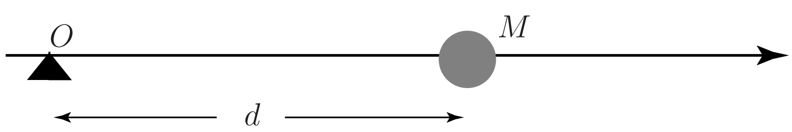

Figure 3. The moment of the mass M about O is Md.

In words, the moment of the mass about O, is the mass multiplied by its distance from O. The units of moment will therefore be kg m if the mass is measured in kilogrammes and the distance in metres; (unless specified otherwise these will be the units we shall always use).

Calculate the moment of the mass about O in each of the following cases.

1. (a)

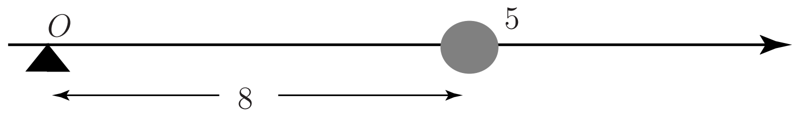

Figure 4.

(b)

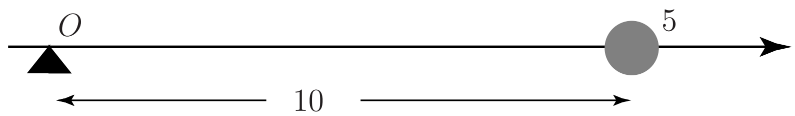

Figure 5.

Your solution

(a) (b)

(a) 40 Kg m (b) 50 Kg m

Intuition tells us that a large moment corresponds to a large turning effect. A mass placed 8 metres from the origin has a smaller turning effect than the same mass placed 10 metres from the origin. This is, of course, why it is easier to rock a see-saw by pushing it at a point further from the pivot. Our intuition also tells us the side of the pivot on which the masses are placed is important. Those placed to the left of the pivot have a turning effect opposite to those placed to the right.

For a collection of masses the moment of the total mass located at the centre of mass is equal to the sum of the moments of the individual masses. This definition enables us to calculate the position of the centre of mass. It is conventional to label the x coordinate of the centre of mass as $\bar{x}$, pronounced 'x bar'.

Key Point

The moment of the total mass located at the centre of mass is equal to the sum of the moments of the individual masses.

Objects of mass m and $3m$ are placed at the locations shown in Figure 6. Find the distance $\bar{x}$ of the centre of mass from the origin O.

Figure 6

First calculate the sum of the individual moments:

Your solution

$(6)(m) + 10(3m) = 36m$

The moment of the total mass about O is $(4m)\bar{x}$.
Now, the moment of the total mass is equal to the sum of the moments of the individual masses. Write down and solve the equation satisfied by $\bar{x}$.

Your solution

$36m = 4m\bar{x}$, so $\bar{x} = 9$

So the centre of mass is located a distance 9 units along the x-axis. Note that it is closer to the position of the $3m$ mass than to the position of the $1m$ mass.

Obtain an equation for the location of the centre of mass of two objects of masses m_1 and m_2 located at *distances* x_1 and x_2 respectively, as shown in Figure 7(a). What happens if the masses are positioned on opposite sides of the origin as shown in Figure 7(b)?

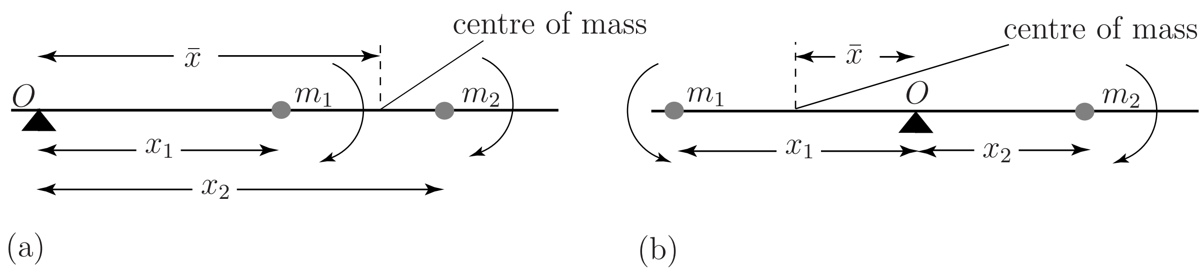

Figure 7

Referring to Figure 7(a), first write down an expression for the sum of the individual moments:

Your solution

$m_1x_1 + m_2x_2$

Write expressions for the total mass and for the moment of the total mass:

Your solution

$m_1 + m_2, \qquad (m_1 + m_2)\bar{x}$

The moment of the total mass is equal to the sum of the moments of the individual masses. Write down and solve the equation satisfied by $\bar{x}$.

Your solution

equation for $\bar{x}$: so $\bar{x} =$

$(m_1 + m_2)\bar{x} = m_1x_1 + m_2x_2 \quad$ so $\quad \bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$

For the second case, as described in Figure 7(b), the mass m_1 positioned on the left-hand side has a turning effect **opposite** to that of the mass m_2 positioned on the right-hand side. To take account of this difference we use a minus sign when determining the moment of m_1 about the origin. This gives a total moment

$$-(m_1x_1) + (m_2x_2)$$

However, this is precisely what would have been obtained if, when working out the moment of a mass we use its *coordinate* (which takes account of sign) rather than using its *distance* from the origin.
The formula obtained in the previous example can be generalised very easily to deal with the situation of n masses, $m_1, m_2, \ldots, m_n$ located at **coordinate positions** $x_1,\ x_2, \ldots x_n$:

Key Point

The centre of mass of $m_1, m_2, \ldots, m_n$ located at $x_1,\ x_2, \ldots x_n$ is

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

Calculate the centre of mass of the 4 masses distributed as shown in Figure 8.

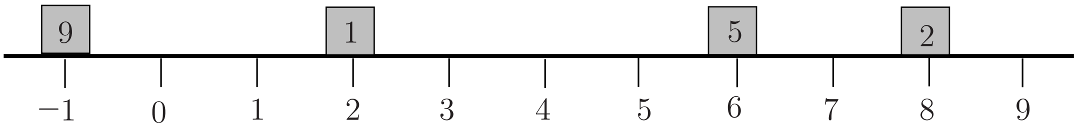

Figure 8.

From the keypoint: $\bar{x} = \dfrac{\sum_{i=1}^{4} m_i x_i}{\sum_{i=1}^{4} m_i}$

Your solution

$\bar{x} =$

$$\bar{x} = \frac{(9)(-1)+(1)(2)+(5)(6)+(2)(8)}{9+1+5+2} = \frac{39}{17}$$

The centre of mass is located a distance $\frac{39}{17} \approx 2.29$ units along the x-axis from O.

Distribution of particles in a plane

If the particles are distributed in a plane then the position of the centre of mass can be calculated in a similar way.

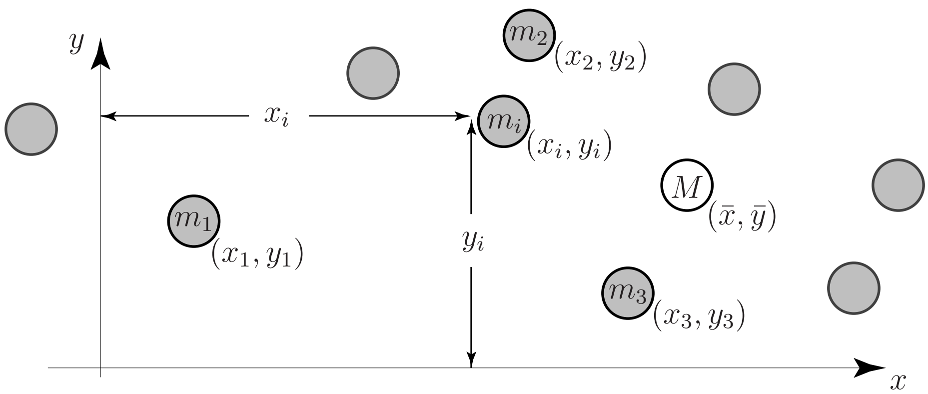

Figure 9. These masses are distributed throughout the xy plane.

Now we must consider the moments of the individual masses taken about the x-axis and about the y-axis. For example, in Figure 9, mass m_i has a moment $m_i y_i$ about the x-axis and a moment $m_i x_i$ about the y-axis. Now we define the centre of mass at that point $(\bar{x}, \bar{y})$ such that the total mass $M = m_1 + m_2 + \ldots m_n$ placed at this point would have the same moment about each of the axes as the sum of the individual moments of the particles about these axes.

Key Point

The centre of mass of $m_1, m_2, \ldots, m_n$ located at (x_1, y_1), $(x_2, y_2), \ldots (x_n, y_n)$ has coordinates

$$\bar{x} = \frac{\displaystyle\sum_{i=1}^{n} m_i x_i}{\displaystyle\sum_{i=1}^{n} m_i}, \qquad \bar{y} = \frac{\displaystyle\sum_{i=1}^{n} m_i y_i}{\displaystyle\sum_{i=1}^{n} m_i}$$

Masses of 5kg, 3kg and 9kg are located at the points with coordinates $(-1, 1)$, $(4, 3)$, and $(8, 7)$ respectively. Find the coordinates of their centre of mass.

Here $n = 3$.

$$\bar{x} = \frac{\displaystyle\sum_{i=1}^{3} m_i x_i}{\displaystyle\sum_{i=1}^{3} m_i} = \frac{5(-1) + 3(4) + 9(8)}{5 + 3 + 9} = \frac{79}{17} \approx 4.65$$

Your solution

$\bar{y} =$

$\bar{y} = \frac{5(1)+3(3)+9(7)}{17} = 4.53$

Hence the centre of mass is located at the point $(4.65, 4.53)$.

Exercises

1. Explain what is meant by the centre of mass of a collection of point masses.
2. Find the x coordinate of the centre of mass of 5 identical masses placed at $x = 2$, $x = 5$, $x = 7$, $x = 9$, $x = 12$.
3. Derive the formula for $\bar{y}$ given in the previous Key Point.

Answers 2. $\bar{x} = 7$

2. Finding the centre of mass of a plane uniform lamina

In the previous section we calculated the centre of mass of several individual point masses. We are now interested in finding the centre of mass of a thin sheet of material, such as a plane sheet of metal. Such an object is called a **lamina**. The mass is not now located at individual points. Rather, it is distributed continuously over the lamina. In what follows we assume that the mass is distributed uniformly over the lamina and you will see how integration as the limit of a sum is used to find the centre of mass.

Figure 10 shows a lamina where the centre of mass has been marked at point G with coordinates $(\bar{x}, \bar{y})$. If the total mass of the lamina is M then the moments about the y and x axes are respectively $M\bar{x}$ and $M\bar{y}$. Our approach to locating the position of G i.e. finding $\bar{x}$ and $\bar{y}$ is to divide the lamina into many small pieces, find the mass of each piece, and the moment of each piece about the axes. The sum of the moments of the individual pieces about the y-axis must then be equal to $M\bar{x}$ and the sum of the moments of the individual pieces about the x-axis to $M\bar{y}$.

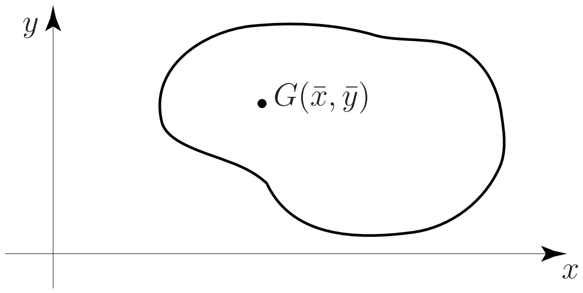

Figure 10. The centre of mass of the lamina is located at $G(\bar{x}, \bar{y})$

There are no formulae which can be memorized for finding the centre of mass of a lamina because of the wide variety of possible shapes. Instead you should be familiar with the general technique for deriving the centre of mass.

An important preliminary concept is 'mass per unit area'.

Mass per unit area

Suppose we have a uniform lamina and select a piece of the lamina which has area equal to one unit. Let ρ stand for the mass of such a piece. Then ρ is called the **mass per unit area**. The mass of any other piece can be expressed in terms of ρ. For example, an area of 2 units must have mass 2ρ an area of 3 units must have mass 3ρ, and so on. Any portion of the lamina which has area A has mass ρA.

Key Point

If a lamina has mass per unit area, ρ, then the mass of part of the lamina having area A is $A\rho$.

We will illustrate the calculation of centre of mass in the following Examples.

Consider the plane sheet, or lamina, shown in Figure 11. Find the location of its centre of mass.

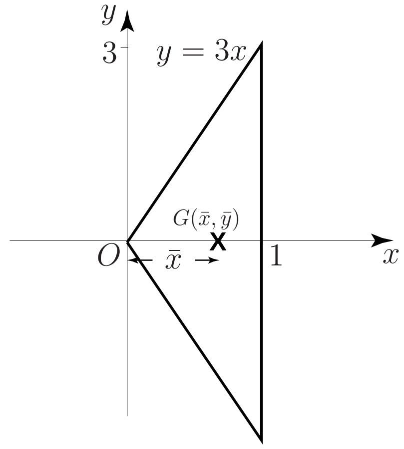

Figure 11. By symmetry the centre of mass of this lamina lies on the x-axis

First inspect the Figure and note the symmetry of the lamina. Purely from the symmetry, what must be the y coordinate, $\bar{y}$, of the centre of mass ?

Your solution

$\bar{y} = 0$ since the centre of mass must lie on the x-axis

Let ρ stand for the mass per unit area of the sheet. The total area is 3 units. The total mass is therefore 3ρ. Its moment about the y-axis is $3\rho\bar{x}$ where $\bar{x}$ is the x coordinate of the centre of mass.

To find $\bar{x}$ first divide the sheet into a large number of thin vertical slices. In Figure 12 a typical slice has been highlighted. Note that the slice has been drawn from the point P on the graph of $y = 3x$. The point P has coordinates (x, y). The thickness of the slice is δx. This notation is consistent with that introduced in Block 1.

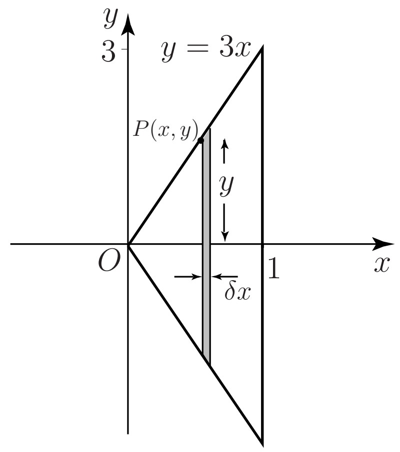

Figure 12. A typical slice of this sheet has been shaded.

Assuming that the slice is rectangular in shape, write down its area.

Your solution

$2y\delta x$

Writing ρ as the mass per unit area, write down the mass of the slice.

Your solution

$(2y\delta x)\rho$

The centre of mass of this slice lies on the x-axis. So the slice can be assumed to be a point mass, $2y\rho\delta x$, located a distance x from O.

Write down the moment of the mass of the slice about the y-axis:

Your solution

$(2y\delta x)\rho x$

By adding up contributions from all such slices in the lamina we obtain the sum of the moments of the individual masses:

$$\sum_{x=0}^{x=1} 2\rho xy\delta x$$

The limits on the sum are chosen so that all slices are included.

Write down the integral defined by letting $\delta x \to 0$:

Your solution

$\int_{x=0}^{x=1} 2\rho xy\,\mathrm{d}x$

Noting that $y = 3x$ express the integrand in terms of x and evaluate it:

Your solution

$\int_0^1 6\rho x^2\,\mathrm{d}x = [2\rho x^3]_0^1 = 2\rho$

This must equal the moment of the total mass acting at the centre of mass:

Hence

$$3\rho\bar{x} = 2\rho \qquad \text{from which} \quad \bar{x} = \frac{2}{3}$$

Now the coordinates of the centre of mass are thus $(\frac{2}{3}, 0)$.

Find the centre of mass of the plane lamina shown in Figure 13.

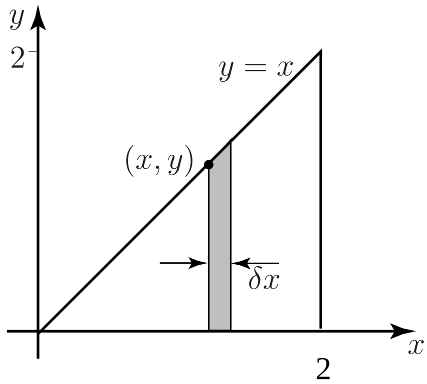

Figure 13.

The coordinates of $\bar{x}$ and $\bar{y}$ must be calculated separately.

To calculate $\bar{x}$

Let ρ equal the mass per unit area. Write down the total area, the total mass and its moment about the y-axis:

Your solution

$2, 2\rho, 2\rho\bar{x}$

To calculate $\bar{x}$ the lamina is divided into thin slices; a typical slice is shown in Figure 13. We assume that the shaded slice is rectangular.

Write down the height of the typical strip shown in Figure 13.

Your solution

y

Write down the area and mass of the typical strip.

Your solution

area = mass =

$y\delta x, \quad (y\delta x)\rho$

Write down the moment about the y-axis of the typical strip.

Your solution

$(y\delta x)\rho x$

The sum of the moments of all strips is

$$\sum_{x=0}^{x=2} \rho x y \delta x$$

Write down the integral which follows as $\delta x \to 0$.

Your solution

$\int_0^2 \rho x y \, \mathrm{d}x$

In this example, $y = x$. Substitute this for y in the integral, and evaluate it.

Your solution

$\int_0^2 \rho x^2 \mathrm{d}x = \frac{8}{3}\rho$

Equating the sum of individual moments and the total moment gives

$$2\rho\bar{x} = \frac{8}{3}\rho \qquad \text{from which} \quad \bar{x} = \frac{4}{3}$$

To calculate $\bar{y}$

We will illustrate two alternative ways of calculating $\bar{y}$.

Referring to Figure 13, the centre of mass of the slice must lie half way along its length, that is its y coordinate is $\frac{y}{2}$. Assume that all the mass of the slice, $y\rho\delta x$, acts at this point. Then its moment about the x-axis is $y\rho\delta x \, \frac{y}{2}$. Adding contributions from all slices gives the sum

$$\sum_{x=0}^{x=2} \frac{y^2\rho}{2}\delta x$$

Write down the integral which is defined as $\delta x \to 0$.

Your solution

$\int_{x=0}^2 \frac{\rho y^2}{2} \mathrm{d}x$

which we can write as

$$\rho \int_{x=0}^{2} \frac{y^2}{2} \mathrm{d}x$$

In this example $y = x$, so the integral becomes

$$\rho \int_{x=0}^{2} \frac{x^2}{2} \mathrm{d}x$$

which equals $\frac{4\rho}{3}$. This is the sum of the individual moments about the x-axis and must equal the moment of the total mass about the x-axis which has already been found as $2\rho\bar{y}$. Therefore

$$2\rho\bar{y} = \frac{4\rho}{3} \qquad \text{from which} \quad \bar{y} = \frac{2}{3}$$

Finally, the coordinates of the centre of mass are $(\frac{4}{3}, \frac{2}{3})$.

Consider now an alternative way of finding $\bar{y}$.

This time the lamina is divided into a number of horizontal slices; a typical slice is shown in Figure 14.

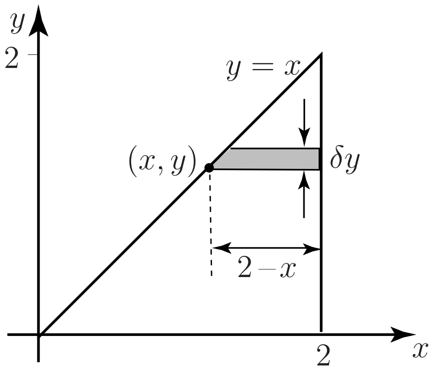

Figure 14. A typical horizontal slice is shaded

The length of the typical slice shown is $2 - x$.

Write down its area, its mass and its moment about the x-axis.

Your solution

$(2-x)\delta y, \qquad \rho(2-x)\delta y, \; \rho(2-x)y\delta y$

Write down the expression for the sum of all such moments and the corresponding integral as $\delta y \to 0$.

Your solution

$\sum_{y=0}^{y=2} \rho(2-x)y\delta y, \qquad \int_0^2 \rho(2-x)y\,\mathrm{d}y$

Now, since $y = x$ the integral can be written entirely in terms of y as

$$\int_0^2 \rho(2-y)y\,\mathrm{d}y$$

Evaluate the integral:

Your solution

$\frac{4\rho}{3}$

As before the total mass is 2ρ, and its moment about the x-axis is $2\rho\bar{y}$. Hence

$$2\rho\bar{y} = \frac{4\rho}{3} \qquad \text{from which} \quad \bar{y} = \frac{2}{3} \qquad \text{as before}$$

Find the position of the centre of mass of a uniform semi-circular lamina of radius a.

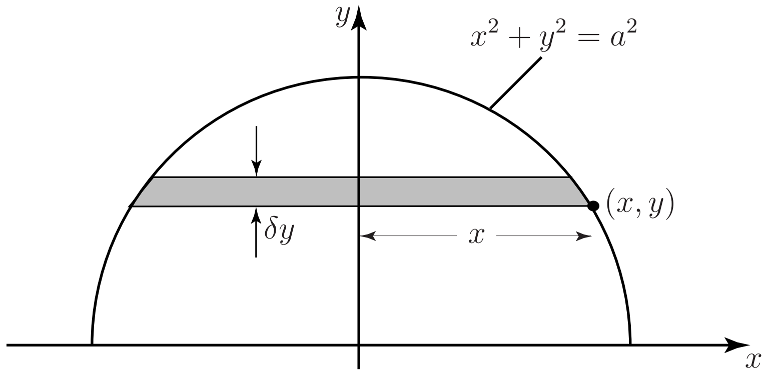

Figure 15. A typical horizontal strip is shaded.

The lamina is shown in Figure 15. The equation of a circle centre the origin, and of radius a is $x^2 + y^2 = a^2$.

By symmetry $\bar{x} = 0$. However it is necessary to calculate $\bar{y}$.

The lamina is divided into a number of horizontal strips and a typical strip is shown. Assume that each strip is rectangular. Writing the mass per unit area as ρ, identify the area and the mass of the strip:

Your solution

$2x\delta y$, $\quad 2x\rho\delta y$

Write down the moment of the mass about the x-axis:

Your solution

$2x\rho y\delta y$

Write down the expression representing the sum of the moments of all strips and the corresponding integral obtained as $\delta y \to 0$.

Your solution

$\sum_{y=0}^{y=a} 2x\rho y\delta y$, $\quad \int_0^a 2x\rho y\,dy$

Now since $x^2 + y^2 = a^2$ we have $x = \sqrt{a^2 - y^2}$ and so the integral becomes:

$$\int_0^a 2\rho y\sqrt{a^2 - y^2}\mathrm{d}y$$

Evaluate this integral by making the substitution $u^2 = a^2 - y^2$ to obtain the total moment.

Your solution

$\frac{2\rho a^3}{3}$

The total area is half that of a circle of radius a, that is $\frac{1}{2}\pi a^2$. The total mass is $\frac{1}{2}\pi a^2\rho$ and its moment is $\frac{1}{2}\pi a^2\rho\bar{y}$.

Hence

$$\frac{1}{2}\pi a^2\rho\bar{y} = \frac{2\rho a^3}{3} \qquad \text{from which} \quad \bar{y} = \frac{4a}{3\pi}$$

Exercises

1. Find the centre of mass of a lamina bounded by $y^2 = 4x$, for $0 \le x \le 9$.

Answers 1. $(\frac{27}{5}, 0)$.