Moment of inertia

15.3

Introduction

In this section we show how integration is used to calculate moments of inertia. These are essential for an understanding of the dynamics of rotating bodies such as fly wheels.

Prerequisites

Before starting this Section you should . . .

① understand integration as the limit of a sum

② be able to calculate definite integrals

Learning Outcomes

After completing this Section you should be able to . . .

✓ calculate the moment of inertia of a number of simple plane bodies

1. Introduction

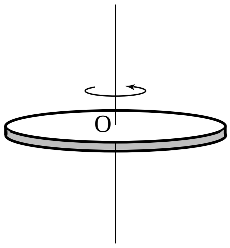

Figure 1. A lamina rotating about an axis through O

Figure 1 shows a lamina, or plane sheet, which is allowed to rotate about an axis perpendicular to the plane of the lamina and through O. The **moment of inertia** about this axis is a measure of how difficult it is to rotate the lamina. It plays the same role for rotating bodies that the mass of an object plays when dealing with motion in a line. An object with large mass needs a large force to achieve a given acceleration. Similarly, an object with large moment of inertia needs a large turning force to achieve a given angular acceleration. Thus knowledge of the moments of inertia of laminas, and also of solid bodies, is essential for understanding their rotational properties.
In this section we show how the idea of integration as the limit of a sum can be used to find the moment of inertia of a lamina.

2. Calculating the moment of inertia

Suppose a lamina is divided into a large number of small pieces or *elements*. A typical element is shown in Figure 2.

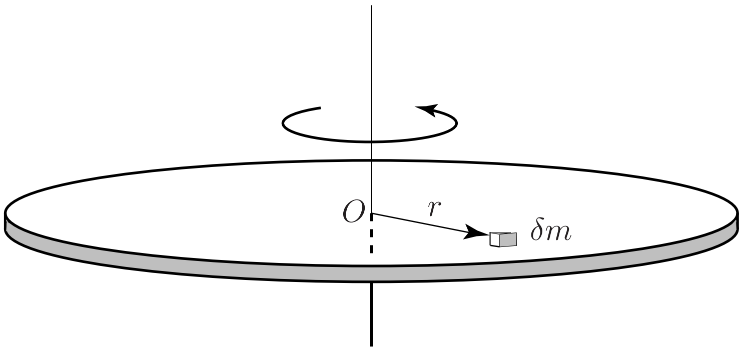

Figure 2. The moment of inertia of the small element is $\delta m\, r^2$

The element has mass δm, and is located a distance r from the axis through O. The moment of inertia of this small piece about the given axis is defined to be $\delta m\, r^2$, that is, the mass multiplied by the square of its distance from the axis of rotation. To find the total moment of inertia we sum the individual contributions to give

$$\sum r^2 \delta m$$

where the sum must be taken in such a way that all parts of the lamina are included. As $\delta m \to 0$ we obtain the following integral as the definition of moment of inertia, I:

Key Point

$$\text{moment of inertia } I = \int r^2 \mathrm{d}m$$

where the limits of integration are chosen so that the entire lamina is included.

The unit of moment of inertia is $\mathrm{kg\,m^2}$.

We shall illustrate how the moment of inertia is actually calculated in practice, in the following examples.

Calculate the moment of inertia about the y-axis of the square lamina of mass M and width b, shown in Figure 3. The moment of inertia about the y-axis is a measure of the resistance to rotation around this axis.

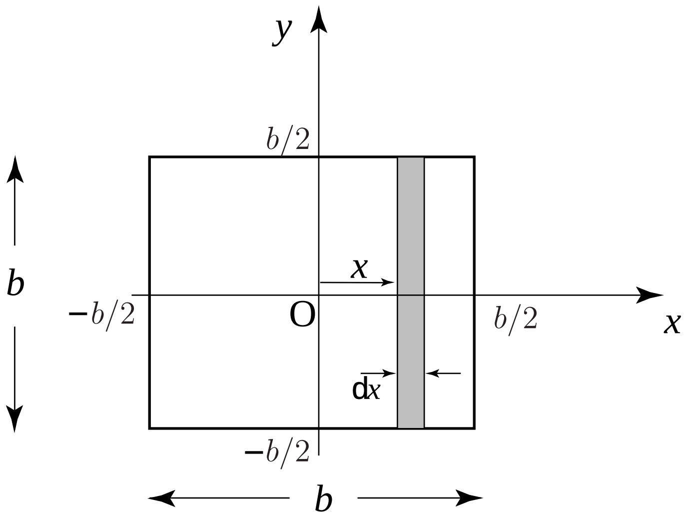

Figure 3. A square lamina rotating about the y-axis

Let the mass per unit area of the lamina be ρ. Then, because its total area is b^2, its total mass M is $b^2\rho$.

Imagine that the lamina has been divided into a large number of thin vertical strips. A typical strip is shown in Figure 3. The strips are chosen in this way because each point on a particular strip is approximately the same distance from the axis of rotation (the y-axis).

Referring to Figure 3 write down the width of each strip:

Your solution

δx

Write down the area of the strip

Your solution

$b\delta x$

With ρ as the mass per unit area write down the mass of the strip

Your solution

$\rho b\delta x$

The distance of the strip from the y-axis is x. Write down its moment of inertia

Your solution

$mbx^2\delta x$

Adding contributions from all strips gives the expression $\sum \rho bx^2\delta x$ where the sum must be such that the entire lamina is included. As $\delta x \to 0$ the sum defines an integral. Write down this integral

Your solution

$I = \int_{-b/2}^{b/2} \rho bx^2 \mathrm{d}x$

Note that the limits on the integral have been chosen so that the whole lamina is included. Then

$$I = \rho b \int_{-b/2}^{b/2} x^2 \mathrm{d}x$$

Your solution

$I =$

$\rho b \left[\frac{x^3}{3}\right]_{-b/2}^{b/2}$

Now evaluate the integral.

Your solution

$I =$

$\frac{\rho b^4}{12}$

Noting that $M = b^2\rho$ then we can write I as $\frac{Mb^2}{12}$.

Find the moment of inertia of a circular disc of mass M and radius a about an axis passing through its centre and perpendicular to the disc. See Figure 4.

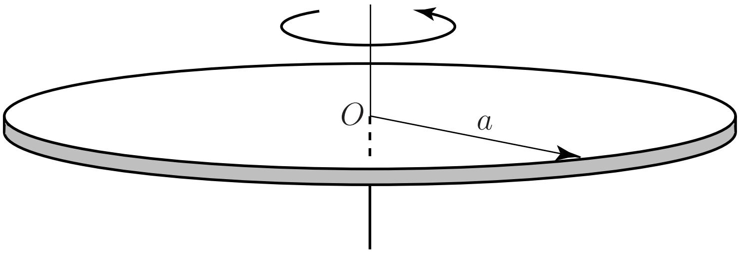

Figure 4. A circular disc rotating about an axis through O

Figure 4 shows the disc lying in the plane of the paper. Imagine that the axis of rotation is coming out of the paper through the centre O and is perpendicular to the disc. The disc can be considered to be spinning in the plane of the paper. Because of the circular symmetry the disc is divided into concentric rings of width δr. A typical ring is shown in Figure 5. Note that each point on the ring is approximately the same distance from the axis of rotation.

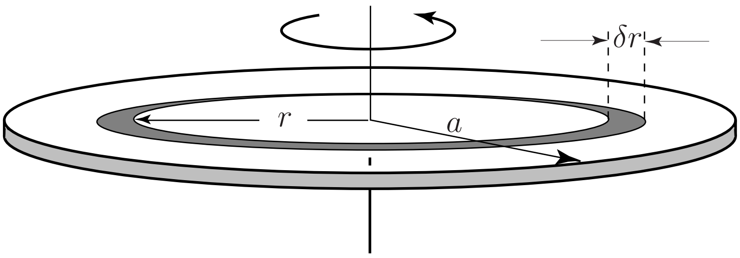

Figure 5. The lamina is divided into many circular rings

The ring has radius r and inner circumference $2\pi r$. Imagine cutting the ring and opening it up. Its area will be approximately that of a long thin rectangle of length $2\pi r$ and width δr. If ρ is the mass per unit area write down an expression for the mass of the ring.

Your solution

$2\pi r\rho\delta r$

The moment of inertia of the ring about O is its mass multiplied by the square of its distance from the axis of rotation. This is $(2\pi r\rho\delta r) \times r^2 = 2\pi r^3\rho\delta r$.

The contribution from all rings must be summed. This gives rise to the sum

$$\sum_{r=0}^{r=a} 2\pi r^3\rho\delta r$$

Note the way that the limits have been chosen so that all rings are included in the sum. As $\delta r \to 0$ the limit of the sum defines the integral

$$\int_0^a 2\rho\pi r^3 \mathrm{d}r$$

Evaluate this integral to give the moment of inertia I.

Your solution

$I = \left[\frac{2\rho\pi r^4}{4}\right]_0^a = \frac{\rho\pi a^4}{2}$

Write down the radius and area of the whole disc.

Your solution

$a, \pi a^2$

With ρ as the mass per unit area, write down the mass of the disc.

Your solution

$\pi a^2 \rho$

But this total mass is M. Hence I can be written $\frac{Ma^2}{2}$.

Exercises

1. The moment of inertia about a diameter of a sphere of radius 1m and mass 1kg is found by evaluating the integral $\frac{3}{8}\int_{-1}^{1}(1-x^2)^2\mathrm{d}x$. Show that this moment of inertia is $\frac{2}{5}\mathrm{kg\,m}^2$.

2. Find the moment of inertia of the square lamina in Figure 3 about one of its sides.

3. Calculate the moment of inertia of a uniform thin rod of mass M and length ℓ about a perpendicular axis of rotation at its end.

4. Calculate the moment of inertia of the rod in Q3 about an axis through its centre and perpendicular to the rod.

5. The **parallel axis theorem** states that the moment of inertia about any axis is equal to the moment of inertia about a parallel axis through the centre of mass, plus the mass of the body $\times$ the square of the distance between the two axes. Verify this theorem for the rod in Q3 and Q4.

6. The **perpendicular axis theorem** applies to a lamina lying in the xy plane. It states that the moment of inertia of the lamina about the z-axis is equal to the sum of the moments of inertia about the x- and y-axes. Suppose that a thin circular disc of mass M and radius a lies in the xy plane and the z axis passes through its centre. The moment of inertia of the disc about this axis is $\frac{1}{2}Ma^2$.

   (a) Use this theorem to find the moment of inertia of the disc about the x and y axes.

   (b) Use the parallel axis theorem to find the moment of inertia of the disc about a tangential axis parallel to the plane of the disc.

Answers

2. $\frac{Mb^2}{3}$.

3. $\frac{1}{3}M\ell^2$.

4. $\frac{1}{12}M\ell^2$.

6. a) The moments of inertia about the x and y axes must be the same by symmetry, and are equal to $\frac{1}{4}Ma^2$.

   b) $\frac{5Ma^2}{4}$.