Infinite Series

16.2

Introduction

We extend the concept of a finite series, met in section 1, to the situation in which the number of terms increase without bound. We define what is meant by an infinite series being convergent by considering the **partial sums** of the series. As prime examples of infinite series we examine the harmonic and the alternating harmonic series and show that the former is divergent and the latter is convergent.
We consider various tests for the convergence of series, in particular we introduce the Ratio test which is a test applicable to series of positive terms. Finally we define the meaning of the terms absolute and conditional convergence.

Prerequisites

Before starting this Section you should . . .

① be able to use the $\sum$ summation notation

② be familiar with the properties of limits

③ be able to use inequalities

Learning Outcomes

After completing this Section you should be able to . . .

✓ use the alternating series test and the ratio test on infinite series

✓ understand the terms *absolute* and *conditional* convergence

1. Introduction

Many of the series considered in section 1 were examples of **finite series** in that they all involved the summation of a finite number of terms. When the number of terms in the series increases without bound we refer to the sum as an **infinite series**. Of particular concern with infinite series is whether they are convergent or divergent. For example, the infinite series

$$1 + 1 + 1 + 1 + \cdots$$

is clearly divergent because the sum of the first n terms increases without bound as more and more terms are taken.
It is less clear as to whether the harmonic and alternating harmonic series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \qquad\qquad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converge or diverge. Indeed you may be surprised to find that the first is divergent and the second is convergent. What we shall do in this section is to consider some simple convergence tests for infinite series. Although we all have an intuitive idea as to the meaning of convergence of an infinite series we must be more precise in our approach. We need a definition for convergence which we can apply rigorously.
First, using an obvious extension of the notation we have used for a finite sum of terms we denote the infinite series:

$$a_1 + a_2 + a_3 + \cdots + a_p + \cdots \qquad \text{by the expression} \quad \sum_{p=1}^{\infty} a_p$$

where a_p is an expression for the p^{th} term in the series. So, as examples:

$$\begin{aligned}
1 + 2 + 3 + \cdots &= \sum_{p=1}^{\infty} p \qquad \text{since the } p^{th} \text{ term is } a_p \equiv p \\
1^2 + 2^2 + 3^2 + \cdots &= \sum_{p=1}^{\infty} p^2 \qquad \text{since the } p^{th} \text{ term is } a_p \equiv p^2 \\
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots &= \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} \qquad \text{here } a_p \equiv \frac{(-1)^{p+1}}{p}
\end{aligned}$$

Consider the infinite series:

$$a_1 + a_2 + \cdots + a_p + \cdots = \sum_{p=1}^{\infty} a_p$$

What we do is to consider the **sequence of partial sums**, $S_1, S_2, \ldots,$ of this series where

$$\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
&\vdots \\
S_n &= a_1 + a_2 + \cdots + a_n
\end{aligned}$$

That is, S_n is the sum of the first n terms of the infinite series. If the limit of the sequence $S_1,\ S_2, \ldots, S_n, \ldots$ can be found; that is

$$\lim_{n\to\infty} S_n = S \qquad \text{(say)}$$

then we define the sum of the infinite series to be S:

$$S = \sum_{p=1}^{\infty} a_p$$

and we say "the series converges to S". Another way of stating this is to say that

$$\sum_{p=1}^{\infty} a_p = \lim_{n\to\infty} \sum_{p=1}^{n} a_p$$

Definition

Convergence of Infinite Series

An infinite series $\sum_{p=1}^{\infty} a_p$ is convergent if the sequence of partial sums

$$S_1,\ S_2,\ S_3,\ \ldots, S_k, \ldots \qquad \text{in which} \quad S_k = \sum_{p=1}^{k} a_p \quad \text{is convergent}$$

Divergence condition for an infinite series

An almost obvious requirement that an infinite series should be convergent is that the individual terms in the series should get smaller and smaller. This leads to the following keypoint:

Key Point

The condition:

$$a_p \to 0 \quad \text{as} \quad p \text{ increases} \quad (\text{mathematically} \quad \lim_{p\to\infty} a_p = 0)$$

is a *necessary* condition for the convergence of the series $\sum_{p=1}^{\infty} a_p$

It is not possible for an infinite series to be convergent unless this condition holds.

Which of the following series cannot be convergent?

(a) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$

(b) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

(c) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

In each case, use the condition from the previous Keypoint.

Your solution

(a) $a_p =$ $\lim_{p\to\infty} a_p =$

$a_p = \frac{p}{p+1}$ $\lim_{p\to\infty} \frac{p}{p+1} = 1$

Hence series is divergent.

Your solution

(b) $a_p =$ $\lim_{p\to\infty} a_p =$

$a_p = \frac{1}{p}$ $\lim_{p\to\infty} a_p = 0$

so this series **may** be convergent. Whether it is or not requires further testing.

Your solution

(c) $a_p =$ $\lim_{p\to\infty} a_p =$

$a_p = \frac{(-1)^{p+1}}{p}$ $\lim_{p\to\infty} a_p = 0$ so again this series **may** be convergent.

Divergence of the harmonic series

The harmonic series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

has a general term $a_n = \frac{1}{n}$ which clearly gets smaller and smaller as $n \to \infty$. However, surprisingly, the series is divergent. Its divergence is demonstrated by showing that the harmonic series is greater than a series which is obviously divergent. We do this by grouping the terms of the harmonic series in a particular way:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \quad \equiv \quad 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$

Now

$$\left(\frac{1}{3}+\frac{1}{4}\right) \quad > \quad \frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$
$$\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) \quad > \quad \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}$$
$$\left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}\right) \quad > \quad \frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}=\frac{1}{2} \text{ etc}$$

and so on. Hence the harmonic series satisfies:

$$1+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) \; + \; \cdots$$
$$> 1 \; + \; \left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)+\cdots$$

The right-hand side of this inequality is clearly divergent so **the harmonic series is divergent**

Convergence of the alternating harmonic series

As with the harmonic series we shall group the terms of the alternating harmonic series, this time to display its convergence.
The alternating harmonic series is:

$$S = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots$$

This series may be re-grouped in two distinct ways.

1st re-grouping

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\cdots = 1-\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{1}{4}-\frac{1}{5}\right)-\left(\frac{1}{6}-\frac{1}{7}\right)\cdots$$

each term in brackets is positive since $\frac{1}{2}>\frac{1}{3}$, $\frac{1}{4}>\frac{1}{5}$ and so on. So we easily conclude that $S<1$ since we are subtracting only positive numbers from 1.

2nd re-grouping

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\cdots = \left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\cdots$$

Again, each term in brackets is positive since $1>\frac{1}{2}$, $\frac{1}{3}>\frac{1}{4}$, $\frac{1}{5}>\frac{1}{6}$ and so on.
So we can also argue that $S>\frac{1}{2}$ since we are adding only positive numbers to the value of the first term, $\frac{1}{2}$. The conclusion that is forced upon us is that

$$\frac{1}{2}<S<1$$

so the alternating series is convergent since its sum, S, lies in the range $\frac{1}{2}\rightarrow 1$. It will be shown in Section 16.5 that $S=\ln 2 \simeq 0.693$

2. General Tests for Convergence

The techniques we have applied to analyse the harmonic and the alternating harmonic series are 'one-off':- they cannot be applied to infinite series in general. However, there are many tests that can be used to determine the convergence properties of infinite series. Of the large number available we shall only consider two such tests in detail.

The alternating series test

An alternating series is a special type of series in which the sign changes from one term to the next. They have the form

$$a_1 - a_2 + a_3 - a_4 + \cdots$$

(in which each $a_i,\ i = 1, 2, 3, \ldots$ is a **positive** number)
Examples are:

(a) $1 - 1 + 1 - 1 + 1 \cdots$ (b) $\frac{1}{3} - \frac{2}{4} + \frac{3}{5} - \frac{4}{6} + \cdots$ (c) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$.

For series of this type there is a simple criterion for convergence:

Key Point

The Alternating Series Test

The alternating series

$$a_1 - a_2 + a_3 - a_4 + \cdots$$

(in which each $a_i,\ i = 1, 2, 3, \ldots$ are **positive** numbers) is convergent if and only if

- the terms continually decrease:

$$a_1 > a_2 > a_3 > \ldots$$

- the terms decrease to zero:

$$a_p \to 0 \quad \text{as } p \text{ increases} \qquad (\text{mathematically } \lim_{p\to\infty} a_p = 0)$$

This is called the **alternating series test**.

Which of the following series are convergent

(a) $\displaystyle\sum_{p=1}^{\infty} (-1)^p \frac{(2p-1)}{(2p+1)}$ (b) $\displaystyle\sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p^2}$

(a) First, write out the series:

Your solution

$-\frac{1}{3} + \frac{3}{5} - \frac{5}{7} + \cdots$

Now examine the series for convergence.

Your solution

$\frac{(2p-1)}{(2p+1)} = \frac{(1-\frac{1}{2p})}{(1+\frac{1}{2p})} \to 1$ as p increases. Since the individual terms of the series do not converge to zero this is therefore a **divergent series**.

(b) Apply the procedure used in (a) to problem (b).

Your solution

This series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ is an alternating series of the form $a_1 - a_2 + a_3 - a_4 + \cdots$ in which $a_p = \frac{1}{p^2}$. The a_p sequence is a decreasing sequence since $1 > \frac{1}{2^2} > \frac{1}{3^2} > \cdots$

Also $\lim_{p\to\infty} \frac{1}{p^2} = 0$. Hence the series is convergent by the alternating series test.

3. The Ratio Test

This test, which is one of the most useful and widely used convergence tests, applies only to series of **positive terms.**

Key Point

The Ratio Test

Let $\displaystyle\sum_{p=1}^{\infty} a_p$ be a series of **positive** terms. Suppose, as p increases, the limit of $\dfrac{a_{p+1}}{a_p}$ equals a number λ. That is $\displaystyle\lim_{p\to\infty} \frac{a_{p+1}}{a_p} = \lambda$. Then, it is possible to show that:

- if $\lambda > 1$, then $\displaystyle\sum_{p=1}^{\infty} a_p$ diverges
- if $\lambda < 1$, then $\displaystyle\sum_{p=1}^{\infty} a_p$ converges
- if $\lambda = 1$, then $\displaystyle\sum_{p=1}^{\infty} a_p$ may be convergent or divergent.

That is, the test is inconclusive in this case.

Example Use the ratio test to examine the convergence of the series

(a) $1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$

(b) $1 + x + x^2 + x^3 + \cdots$

Solution

(i) The general term in this series is $\frac{1}{p!}$ i.e.

$$1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = \sum_{p=1}^{\infty} \frac{1}{p!} \qquad a_p = \frac{1}{p!} \qquad \therefore \qquad a_{p+1} = \frac{1}{(p+1)!}$$

and the ratio

$$\frac{a_{p+1}}{a_p} = \frac{p!}{(p+1)!} = \frac{p(p-1)\ldots(3)(2)(1)}{(p+1)p(p-1)\ldots(3)(2)(1)} = \frac{1}{(p+1)}$$

Solution (contd.)

$$\therefore \qquad \lim_{p\to\infty}\frac{a_{p+1}}{a_p} = \lim_{p\to\infty}\frac{1}{(p+1)} = 0$$

Since $0 < 1$ the series is convergent. In fact, it will be easily shown using the techniques outlined in in Section 16.5 that

$$1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = \mathrm{e} - 1 \approx 1.718$$

(ii) Here we must assume that $x > 0$ since we can only apply the ratio test to a series of positive terms.

Now

$$1 + x + x^2 + x^3 + \cdots \quad = \sum_{p=1}^{\infty} x^{p-1}$$

so that

$$a_p = x^{p-1}\ , \quad a_{p+1} = x^p$$

and

$$\lim_{p\to\infty}\frac{a_{p+1}}{a_p} = \lim_{p\to\infty}\frac{x^p}{x^{p-1}} = \lim_{p\to\infty} x = x$$

Thus, using the ratio test we deduce that (if x is a positive number) this series will only converge if $x < 1$. We will see in Section 16.4 that

$$1 + x + x^2 + x^3 + \cdots \quad = \frac{1}{1-x} \qquad \text{provided} \quad 0 < x < 1.$$

(replace x by $-x$ and choose $p = -1$ in the Binomial series).

Use the ratio test to examine the convergence of the series:

$$\frac{1}{\ln 3} + \frac{8}{(\ln 3)^2} + \frac{27}{(\ln 3)^3} + \cdots$$

First, find the general term of the series.

Your solution

$a_p =$

$$\frac{1}{\ln 3} + \frac{8}{(\ln 3)^2} + \cdots = \sum_{p=1}^{\infty}\frac{p^3}{(\ln 3)^p} \quad \text{so} \quad a_p = \frac{p^3}{(\ln 3)^p}$$

Now find a_{p+1}

Your solution

$a_{p+1} =$

$$a_{p+1} = \frac{(p+1)^3}{(\ln 3)^{p+1}}$$

Finally, obtain $\displaystyle\lim_{p\to\infty}\frac{a_{p+1}}{a_p}$

Your solution

$$\frac{a_{p+1}}{a_p} = \qquad\qquad \therefore \qquad \lim_{p\to\infty}\frac{a_{p+1}}{a_p} =$$

$\displaystyle\frac{a_{p+1}}{a_p} = \left(\frac{p+1}{p}\right)^3\frac{1}{(\ln 3)}$. Now $\displaystyle\left(\frac{p+1}{p}\right)^3 = \left(1+\frac{1}{p}\right)^3 \to 1$ as p increases $\quad\therefore\quad \displaystyle\lim_{p\to\infty}\frac{a_{p+1}}{a_p} = \frac{1}{(\ln 3)} < 1$

Hence **this is a convergent series.**

Note that in all of these examples and guided exercises we have decided upon the convergence or divergence of various series; we have not been able to use the tests to discover what actual number the convergent series converges to.

4. Absolute and Conditional Convergence

The ratio test applies to series of positive terms. Indeed this is true of many related tests for convergence. However, as we have seen, not all series are series of positive terms. To apply the ratio test such series must first be converted into series of positive terms. This is easily done. Consider two series $\displaystyle\sum_{p=1}^{\infty} a_p$ and $\displaystyle\sum_{p=1}^{\infty} |a_p|$. The latter series, obviously directly related to the first, is a series of positive terms.

Using imprecise language, it is harder for the second series to converge than it is for the first, since, in the first, some of the terms may be negative and cancel out part of the contribution from the positive terms. No such cancellations can take place in the second series since they are all positive terms. Thus it is plausible that if $\displaystyle\sum_{p=1}^{\infty} |a_p|$ converges so does $\displaystyle\sum_{p=1}^{\infty} a_p$. This leads to the following definition.

Definition

Conditional Convergence

A convergent series $\displaystyle\sum_{p=1}^{\infty} a_p$ for which its related series $\displaystyle\sum_{p=1}^{\infty} |a_p|$ is divergent is said to be **conditionally convergent**

Absolute Convergence

A convergent series $\displaystyle\sum_{p=1}^{\infty} a_p$ is said to be **absolutely convergent** if $\displaystyle\sum_{p=1}^{\infty} |a_p|$ is convergent.

For example, the alternating harmonic series:

$$\sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is **conditionally convergent** since the series of positive terms

$$\sum_{p=1}^{\infty} \left| \frac{(-1)^{p+1}}{p} \right| \equiv \sum_{p=1}^{\infty} \frac{1}{p} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

is divergent.

Show that the series $-\frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots$ is absolutely convergent.

First, find the general term of the series

Your solution

$-\frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots \quad = \sum_{p=1}^{\infty} (\qquad) \qquad \therefore \qquad a_p \equiv$

$-\frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots \quad = \sum_{p=1}^{\infty} \frac{(-1)^p}{(2p)!} \qquad \therefore \qquad a_p \equiv \frac{(-1)^p}{(2p)!}$

The related series of positive terms is

Your solution

$\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots \quad = \sum_{p=1}^{\infty} (\qquad) \qquad \therefore \qquad a_p =$

$\sum_{p=1}^{\infty} \frac{1}{(2p)!} \qquad$ so $\qquad a_p = \frac{1}{(2p)!}$

Now use the ratio test to examine the convergence of this series

Your solution

p^{th} term $=$ $\qquad\qquad$ $(p+1)^{th}$ term $=$

p^{th} term $= \frac{1}{(2p)!}$ $\qquad\qquad$ $(p+1)^{th}$ term $= \frac{1}{(2(p+1))!}$

What is $\lim_{p \to \infty} \left[\frac{(p+1)^{th} \text{ term}}{p^{th} \text{ term}} \right]$?

Your solution

$$\lim_{p \to \infty} \left[\frac{(p+1)^{th} \text{ term}}{p^{th} \text{ term}} \right] =$$

$$\frac{(2p)!}{(2(p+1))!} = \frac{2p(2p-1)\dots}{(2p+2)(2p+1)2p(2p-1)\dots} = \frac{1}{(2p+2)(2p+1)} \to 0 \text{ as } p \text{ increases.}$$

So the series of positive terms is convergent by the ratio test. Hence $\sum_{p=1}^{\infty} \frac{(-1)^p}{(2p)!}$ is **absolutely convergent.**

Exercises

1. Which of the following alternating series are convergent?

(a) $\sum_{p=1}^{\infty} \frac{(-1)^p \ln(3)}{p}$ (b) $\sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p^2+1}$ (c) $\sum_{p=1}^{\infty} \frac{p\sin(2p+1)\frac{\pi}{2}}{(p+100)}$

2. Use the ratio test to examine the convergence of the series:

(a) $\sum_{p=1}^{\infty} \frac{e^4}{(2p+1)^{p+1}}$ (b) $\sum_{p=1}^{\infty} \frac{p^3}{p!}$ (c) $\sum_{p=1}^{\infty} \frac{1}{\sqrt{p}}$

(d) $\sum_{p=1}^{\infty} \frac{1}{(0.3)^p}$ (e) $\sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{3^p}$

3. For what values of x are the following series absolutely convergent?

(a) $\sum_{p=1}^{\infty} \frac{(-1)^p x^p}{p}$ (b) $\sum_{p=1}^{\infty} \frac{(-1)^p x^p}{p!}$

Answers 1. (a) convergent, (b) convergent, (c) divergent

2. (a) $\lambda = 0$ so convergent, (b) $\lambda = 0$ so convergent, (c) $\lambda = 1$ so test is inconclusive. However, since $\frac{1}{p^{1/2}} > \frac{1}{p}$ then the given series is divergent by comparison with the harmonic series. (d) $\lambda = 10/3$ so divergent, (e) Not a series of positive terms so the ratio test cannot be applied.

3. (a) The related series of positive terms is $\sum_{p=1}^{\infty} \frac{|x|^p}{p}$. For this series, using the ratio test we find $\lambda = |x|$ so the original series is absolutely convergent if $|x| < 1$.

(b) The related series of positive terms is $\sum_{p=1}^{\infty} \frac{|x|^p}{p!}$. For this series, using the ratio test we find $\lambda = 0$ (irrespective of the value of x) so the original series is absolutely convergent for **all** values of x.