Power Series

16.4

Introduction

In this section we consider power series. These are examples of infinite series where each term contains a variable, x, raised to a positive integer power. We use the ratio test to obtain the **radius of convergence** R, of the power series and state the important result that the series is absolutely convergent if $|x| < R$, divergent if $|x| > R$ and may or may not be convergent if $x = \pm R$. Finally, we extend the work to apply to general power series when the variable x is replaced by $(x - x_0)$.

Prerequisites

Before starting this Section you should ...

① have knowledge of infinite series and of the ratio test

② have knowledge of inequalities and of the factorial notation.

Learning Outcomes

After completing this Section you should be able to ...

✓ understand what a power series is

✓ obtain the radius of convergence for a power series

✓ know what a general power series is

1. Power Series

A power series is simply a sum of terms each of which contains a variable raised to a positive integer power. To illustrate:

$$x - x^3 + x^5 - x^7 + \cdots$$

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

are examples of power series. Of course, in section 16.3, we encountered an important example of a power series, viz the binomial series:

$$1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

which, as we have already noted, represents the function $(1+x)^p$ as long as the variable x satisfies $|x| < 1$.

A power series has the general form

$$b_0 + b_1x + b_2x^2 + \cdots = \sum_{p=0}^{\infty} b_px^p$$

where $b_0, b_1, b_2, \cdots$ are constants. Note that, in the summation notation, we have chosen to start the series at $p = 0$. This is to ensure that the power series can include a constant term b_0 since $x^0 = 1$.

The convergence, or otherwise, of a power series, clearly depends upon the value of x chosen. For example, the power series

$$1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots$$

is convergent if $x = -1$ (for then it is the alternating harmonic series) and divergent if $x = +1$ (for then it is the harmonic series).

2. The Radius of Convergence

The most important statement one can make about a power series is that there exists a number, R, called the radius of convergence, such that if $|x| < R$ the power series is absolutely convergent and if $|x| > R$ the power series is divergent. The relation $|x| < R$ is equivalent to

$$-R < x < R.$$

At the two points $x = -R$ and $x = R$ the power series may be convergent or divergent.

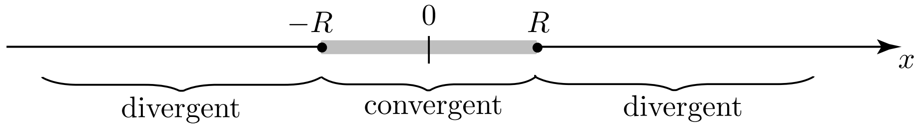
Key Point

Convergence of Power Series

For a power series $\sum_{p=0}^{\infty} b_p x^p$ with radius of convergence R then

- the series converges absolutely if $|x| < R$
- the series diverges if $|x| > R$
- the series may be convergent or divergent at $x = \pm R$

See the following diagram.



For any particular power series $\sum_{p=0}^{\infty} b_p x^p$ the value of R can be obtained using the ratio test. We know, from the ratio test that $\sum_{p=0}^{\infty} b_p x^p$ is absolutely convergent if

$$\lim_{p\to\infty} \frac{|b_{p+1}x^{p+1}|}{|b_p x^p|} = \lim_{p\to\infty} \left|\frac{b_{p+1}}{b_p}\right| |x| < 1 \qquad \text{implying} \qquad |x| < \lim_{p\to\infty} \left|\frac{b_p}{b_{p+1}}\right| \quad \text{and so} \quad R = \lim_{p\to\infty} \left|\frac{b_p}{b_{p+1}}\right|.$$

Example Find the radius of convergence of the series

$$1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots$$

Solution

Here $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots \quad = \sum_{p=0}^{\infty} \frac{x^p}{p+1}$

so

$$b_p = \frac{1}{p+1} \qquad \therefore \qquad b_{p+1} = \frac{1}{p+2}$$

In this case,

$$R = \lim_{p\to\infty} \left|\frac{p+2}{p+1}\right| = 1$$

so the given series is absolutely convergent if $|x| < 1$ and is divergent if $|x| > 1$.

What happens at the end-points $x=-1$, $x=+1$ of the region of absolute convergence?
At $x=+1$ the series is $1+\frac{1}{2}+\frac{1}{3}+\cdots$ which is divergent (the harmonic series). However, at $x=-1$ the series is $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ which is convergent (the alternating harmonic series).
Finally, therefore, the series

$$1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\cdots$$

is convergent if $-1\leq x<1$.

Find those values of x for which the following power series converges:

$$1+\frac{x}{3}+\frac{x^2}{3^2}+\frac{x^3}{3^3}+\cdots$$

First find the coefficient of x^p.

Your solution

$b_p=$

$b_p=\frac{1}{3^p}$

Now find R, the radius of convergence

Your solution

$$R=\lim_{p\to\infty}\left|\frac{b_p}{b_{p+1}}\right|=$$

$$R=\lim_{p\to\infty}\left|\frac{b_p}{b_{p+1}}\right|=\lim_{p\to\infty}\left|\frac{3^{p+1}}{3^p}\right|=\lim_{p\to\infty}(3)=3.$$

When $x=\pm3$ the series is clearly divergent. Hence the series is convergent only if $-3<x<3$.

3. Properties of Power Series

Let P_1 and P_2 represent two power series with radii of convergence R_1 and R_2 respectively. We can combine P_1 and P_2 together by addition and multiplication. We find

Key Point

The sum (P_1+P_2) and the product (P_1P_2) are each power series with the radius of convergence being the **smaller** of R_1 and R_2.

Power series can also be differentiated and integrated on a term by term basis:

Key Point

$$\frac{\mathrm{d}}{\mathrm{d}x}(P_1) \quad \text{and} \quad \int (P_1)\mathrm{d}x \quad \text{are each power series with radius of convergence } R_1$$

Example We have already seen that

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots \qquad |x| < 1$$

Choose $p = \frac{1}{2}$ and then, by differentiating, obtain the power series expression for $(1+x)^{-\frac{1}{2}}$.

Solution

$$(1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \cdots$$

$\therefore$ differentiating both sides

$$\frac{1}{2}(1+x)^{-\frac{1}{2}} = \frac{1}{2} + \frac{1}{2}\left(-\frac{1}{2}\right)x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}x^2 + \cdots$$

Multiplying through by 2:

$$(1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}x^2 + \cdots$$

which can of course be obtained directly from the expansion for $(1+x)^p$ with $p = -\frac{1}{2}$.

Using the known result that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \qquad |x| < 1$$

find an expression for $\ln(1+x)$. Use this expression to obtain an approximation to $\ln(1.1)$

Integrate both sides of $\frac{1}{1+x} = 1 - x + x^2 - \cdots$

Your solution

$$\int \frac{\mathrm{d}x}{1+x} =$$

$$\int (1 - x + x^2 - \cdots)\mathrm{d}x =$$

$\displaystyle\int \frac{\mathrm{d}x}{1+x} = \ln(1+x) + c$ where c is a constant of integration

$\displaystyle\int (1 - x + x^2 - \cdots)\mathrm{d}x = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + k$ where k is a constant of integration

so we conclude

$$\ln(1+x) + c = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + k \qquad \text{if} \quad |x| < 1$$

Choosing $x = 0$ shows that $c = k$ so they cancel from this equation. Now choose $x = 0.1$ to find $\ln(1 + 0.1)$

Your solution

$\ln(1.1) = 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \cdots \simeq$

$\ln(1.1) \simeq 0.0953$ which is easily checked on your calculator.

4. General Power Series

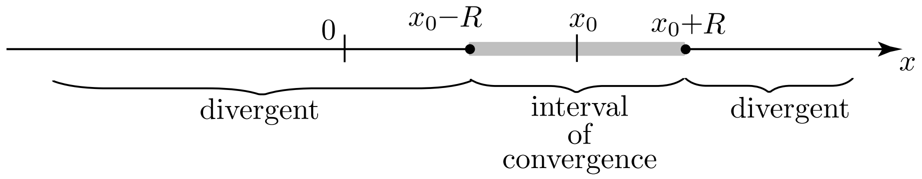
A general power series has the form

$$b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + \cdots = \sum_{p=0}^{\infty} b_p (x - x_0)^p$$

Exactly the same considerations apply to this general power series as apply to the 'special' series $\displaystyle\sum_{p=0}^{\infty} b_p x^p$ except that the variable x is replaced by $(x - x_0)$. The radius of convergence of the general series is obtained in the same way:

$$R = \lim_{p\to\infty} \left| \frac{b_p}{b_{p+1}} \right|$$

and the interval of convergence is now shifted to have centre at $x = x_0$ (see diagram below). The series is absolutely convergent if $|x - x_0| < R$, diverges if $|x - x_0| > R$ and may or may not converge if $|x - x_0| = R$.



Find the radius of convergence of the series

$$1-(x-1)+(x-1)^2-(x-1)^3+\cdots$$

First find an expression for the general term

Your solution

$$1-(x-1)+(x-1)^2-(x-1)^3+\cdots=\sum_{p=0}^{\infty}(\qquad\qquad)$$

$$\sum_{p=0}^{\infty}(x-1)^p(-1)^p \qquad \text{so} \qquad b_p=(-1)^p$$

Now obtain the radius of convergence

Your solution

$$\lim_{p\to\infty}\left|\frac{b_p}{b_{p+1}}\right|= \qquad\qquad \therefore \qquad\qquad R=$$

$$\lim_{p\to\infty}\left|\frac{b_p}{b_{p+1}}\right|=\lim_{p\to\infty}\left|\frac{(-1)^p}{(-1)^{p+1}}\right|=1.$$

Hence $R=1$, so the series is absolutely convergent if $|x-1|<1$

Finally, decide on the convergence at $|x-1|=1$ (i.e. at $x-1=-1$ and $x-1=1$ i.e. $x=0$ and $x=2$). At $x=0$ the series is $1+1+1+\cdots$ which diverges and at $x=2$ the series is $1-1+1-1\cdots$ which also diverges. Thus the given series only converges if $|x-1|<1$ i.e. $0<x<2$.