

Power Series

16.4



Introduction

In this section we consider power series. These are examples of infinite series where each term contains a variable, x , raised to a positive integer power. We use the ratio test to obtain the **radius of convergence** R , of the power series and state the important result that the series is absolutely convergent if $|x| < R$, divergent if $|x| > R$ and may or may not be convergent if $x = \pm R$. Finally, we extend the work to apply to general power series when the variable x is replaced by $(x - x_0)$.



Prerequisites

Before starting this Section you should ...

- ① have knowledge of infinite series and of the ratio test
- ② have knowledge of inequalities and of the factorial notation.



Learning Outcomes

After completing this Section you should be able to ...

- ✓ understand what a power series is
- ✓ obtain the radius of convergence for a power series
- ✓ know what a general power series is

1. Power Series

A power series is simply a sum of terms each of which contains a variable raised to a positive integer power. To illustrate:

$$x - x^3 + x^5 - x^7 + \dots$$

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

are examples of power series. Of course, in section 16.3, we encountered an important example of a power series, viz the binomial series:

$$1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots$$

which, as we have already noted, represents the function $(1+x)^p$ as long as the variable x satisfies $|x| < 1$.

A power series has the general form

$$b_0 + b_1x + b_2x^2 + \dots = \sum_{p=0}^{\infty} b_p x^p$$

where b_0, b_1, b_2, \dots are constants. Note that, in the summation notation, we have chosen to start the series at $p = 0$. This is to ensure that the power series can include a constant term b_0 since $x^0 = 1$.

The convergence, or otherwise, of a power series, clearly depends upon the value of x chosen. For example, the power series

$$1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \dots$$

is convergent if $x = -1$ (for then it is the alternating harmonic series) and divergent if $x = +1$ (for then it is the harmonic series).

2. The Radius of Convergence

The most important statement one can make about a power series is that there exists a number, R , called the radius of convergence, such that if $|x| < R$ the power series is absolutely convergent and if $|x| > R$ the power series is divergent. The relation $|x| < R$ is equivalent to

$$-R < x < R.$$

At the two points $x = -R$ and $x = R$ the power series may be convergent or divergent.



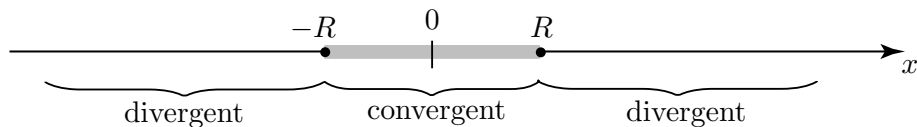
Key Point

Convergence of Power Series

For a power series $\sum_{p=0}^{\infty} b_p x^p$ with radius of convergence R then

- the series converges absolutely if $|x| < R$
- the series diverges if $|x| > R$
- the series may be convergent or divergent at $x = \pm R$

See the following diagram.



For any particular power series $\sum_{p=0}^{\infty} b_p x^p$ the value of R can be obtained using the ratio test. We know, from the ratio test that $\sum_{p=0}^{\infty} b_p x^p$ is absolutely convergent if

$$\lim_{p \rightarrow \infty} \frac{|b_{p+1} x^{p+1}|}{|b_p x^p|} = \lim_{p \rightarrow \infty} \left| \frac{b_{p+1}}{b_p} \right| |x| < 1 \quad \text{implying} \quad |x| < \lim_{p \rightarrow \infty} \left| \frac{b_p}{b_{p+1}} \right| \quad \text{and so} \quad R = \lim_{p \rightarrow \infty} \left| \frac{b_p}{b_{p+1}} \right|.$$

Example Find the radius of convergence of the series

$$1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \dots$$

Solution

Here $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \dots = \sum_{p=0}^{\infty} \frac{x^p}{p+1}$

so

$$b_p = \frac{1}{p+1} \quad \therefore \quad b_{p+1} = \frac{1}{p+2}$$

In this case,

$$R = \lim_{p \rightarrow \infty} \left| \frac{p+2}{p+1} \right| = 1$$

so the given series is absolutely convergent if $|x| < 1$ and is divergent if $|x| > 1$.

What happens at the end-points $x = -1$, $x = +1$ of the region of absolute convergence?

At $x = +1$ the series is $1 + \frac{1}{2} + \frac{1}{3} + \dots$ which is divergent (the harmonic series). However, at $x = -1$ the series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ which is convergent (the alternating harmonic series). Finally, therefore, the series

$$1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \dots$$

is convergent if $-1 \leq x < 1$.



Find those values of x for which the following power series converges:

$$1 + \frac{x}{3} + \frac{x^2}{3^2} + \frac{x^3}{3^3} + \dots$$

First find the coefficient of x^p .

Your solution

$$b_p =$$

$$\frac{d^p}{p!} = d^p$$

Now find R , the radius of convergence

Your solution

$$R = \lim_{p \rightarrow \infty} \left| \frac{b_p}{b_{p+1}} \right| =$$

When $x = \pm 3$ the series is clearly divergent. Hence the series is convergent only if $-3 < x < 3$.

$$R = \lim_{p \rightarrow \infty} \left| \frac{3^{p+1}}{3^p} \right| = \lim_{p \rightarrow \infty} 3 = 3.$$

3. Properties of Power Series

Let P_1 and P_2 represent two power series with radii of convergence R_1 and R_2 respectively. We can combine P_1 and P_2 together by addition and multiplication. We find



Key Point

The sum $(P_1 + P_2)$ and the product $(P_1 P_2)$ are each power series with the radius of convergence being the **smaller** of R_1 and R_2 .

Power series can also be differentiated and integrated on a term by term basis:



Key Point

$\frac{d}{dx}(P_1)$ and $\int (P_1)dx$ are each power series with radius of convergence R_1

Example We have already seen that

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \dots \quad |x| < 1$$

Choose $p = \frac{1}{2}$ and then, by differentiating, obtain the power series expression for $(1+x)^{-\frac{1}{2}}$.

Solution

$$(1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3 + \dots$$

\therefore differentiating both sides

$$\frac{1}{2}(1+x)^{-\frac{1}{2}} = \frac{1}{2} + \frac{1}{2} \left(-\frac{1}{2}\right)x + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{2}x^2 + \dots$$

Multiplying through by 2:

$$(1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}x^2 + \dots$$

which can of course be obtained directly from the expansion for $(1+x)^p$ with $p = -\frac{1}{2}$.



Using the known result that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \quad |x| < 1$$

find an expression for $\ln(1+x)$. Use this expression to obtain an approximation to $\ln(1.1)$

Integrate both sides of $\frac{1}{1+x} = 1 - x + x^2 - \dots$

Your solution

$$\int \frac{dx}{1+x} =$$

$$\int (1 - x + x^2 - \dots) dx =$$

$$\ln(1+x) + c = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + k \quad \text{if } |x| < 1$$

so we conclude

$$\ln(1+x) + c = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + k \quad \text{if } |x| < 1$$

Choosing $x = 0$ shows that $c = k$ so they cancel from this equation. Now choose $x = 0.1$ to find $\ln(1 + 0.1)$

Your solution

$$\ln(1.1) = 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \dots \simeq$$

$$\ln(1.1) \simeq 0.0953 \text{ which is easily checked on your calculator.}$$

4. General Power Series

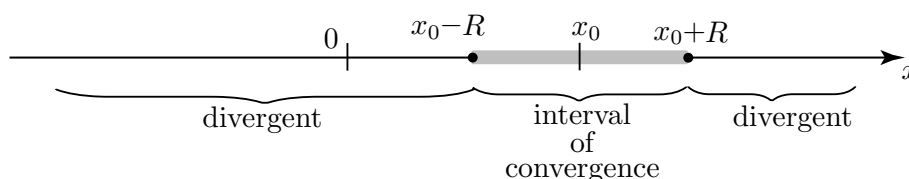
A general power series has the form

$$b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + \dots = \sum_{p=0}^{\infty} b_p(x - x_0)^p$$

Exactly the same considerations apply to this general power series as apply to the 'special' series $\sum_{p=0}^{\infty} b_p x^p$ except that the variable x is replaced by $(x - x_0)$. The radius of convergence of the general series is obtained in the same way:

$$R = \lim_{p \rightarrow \infty} \left| \frac{b_p}{b_{p+1}} \right|$$

and the interval of convergence is now shifted to have centre at $x = x_0$ (see diagram below). The series is absolutely convergent if $|x - x_0| < R$, diverges if $|x - x_0| > R$ and may or may not converge if $|x - x_0| = R$.





Find the radius of convergence of the series

$$1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + \dots$$

First find an expression for the general term

Your solution

$$1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + \dots = \sum_{p=0}^{\infty} (\quad)$$

$$d(1-) = {}^d q \quad \text{os} \quad d(1-) {}_d(1-x) \sum_{\infty}^{0=d}$$

Now obtain the radius of convergence

Your solution

$$\lim_{p \rightarrow \infty} \left| \frac{b_p}{b_{p+1}} \right| = \quad \therefore \quad R =$$

$$\lim_{\infty \leftarrow d} \left| \frac{b^d}{b^{d+1}} \right| = \lim_{\infty \leftarrow d} \left| \frac{(-1)^d}{1+d(1-)} \right| = 1. \quad \text{Hence } R = 1, \text{ so the series is absolutely convergent if } |x - 1| < 1$$

Finally, decide on the convergence at $|x - 1| = 1$ (i.e. at $x - 1 = -1$ and $x - 1 = 1$ i.e. $x = 0$ and $x = 2$). At $x = 0$ the series is $1 + 1 + 1 + \dots$ which diverges and at $x = 2$ the series is $1 - 1 + 1 - 1 \dots$ which also diverges. Thus the given series only converges if $|x - 1| < 1$ i.e. $0 < x < 2$.

