Maclaurin and Taylor Series

16.5

Introduction

In this Section we examine how functions may be expressed in terms of power series. This is an extremely useful way of expressing a function since (as we shall see) we can then replace 'complicated' functions in terms of 'simple' polynomials. The only requirement (of any significance) is that the 'complicated' function should be *smooth*; this means that at a point of interest, it must be possible to differentiate the function as often as we please.

Prerequisites

Before starting this Section you should . . .

① have knowledge of power series and of the ratio test

② be able to differentiate simple functions

③ be familiar with the rules for combining power series

Learning Outcomes

After completing this Section you should be able to . . .

✓ find the Maclaurin and Taylor series expansions of given functions

✓ find Maclaurin expansions of functions by combining known power series together

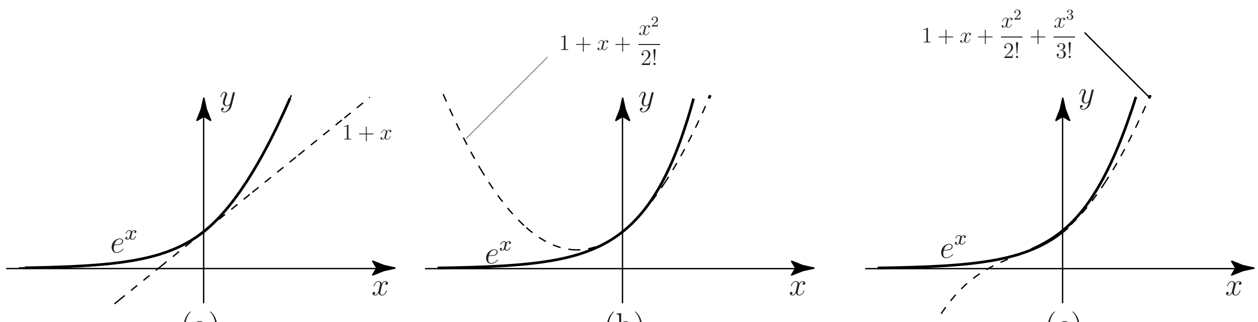
✓ find Maclaurin expansions by using differentiation and integration

1. Maclaurin and Taylor Series

As we shall see, many functions can be represented by power series. In fact we have already seen in earlier Sections examples of such a representation. For example,

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + \cdots \qquad |x| < 1 \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \qquad -1 < x \leq 1 \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{all } x
\end{aligned}$$

The first two examples show that, as long as we constrain x to lie within the domain $|x| < 1$ (or, equivalently, $-1 < x < 1$), then in the first case $\frac{1}{1-x}$ has the **same numerical value** as $1+x+x^2+\cdots$ and in the second case $\ln(1+x)$ has the same numerical value as $x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$. In the third example we see that e^x has the same numerical value as $1 + x + \frac{x^2}{2!} + \cdots$ but in this case there is no restriction to be placed on the value of x since **this** power series converges for all values of x. The following diagram shows this situation geometrically. As more and more terms are used from the series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots$ the curve representing e^x is better and better approximated. In (a) we show the linear approximation to e^x. In (b) and (c) we show, respectively, the quadratic and cubic approximations.



These power series representations are extremely important, from many points of view. Numerically, we can simply replace the function $\frac{1}{1-x}$ by the quadratic expression $1 + x + x^2$ as long as x is so small that powers of x greater than or equal to 3 can be ignored in comparison to quadratic terms. This approach can be used to approximate more complicated functions in terms of simpler polynomials. Our aim now is to see how these power series expansions are obtained.

2. The Maclaurin Series

Consider a function $f(x)$ which can be differentiated at $x = 0$ as often as we please. For example $e^x, \cos x, \sin x$ would fit into this category but $|x|$ would not.
Let us assume that $f(x)$ can be represented by a power series in x:

$$f(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \cdots = \sum_{p=0}^{\infty} b_p x^p$$

where $b_0, b_1, b_2, \ldots$ are constants to be determined.
If we substitute $x = 0$ then, clearly

$$f(0) = b_0$$

The other constants can be determined by further differentiating and, on each differentiation, substituting $x = 0$. For example, differentiating once:

$$f'(x) = 0 + b_1 + 2b_2x + 3b_3x^2 + 4b_4x^3 + \cdots$$

so, putting $x = 0$, we have $f'(0) = b_1$.
Continuing to differentiate:

$$f''(x) = 0 + 2b_2 + 3(2)b_3x + 4(3)b_4x^2 + \cdots$$

so

$$f''(0) = 2b_2 \qquad \text{or} \qquad b_2 = \frac{1}{2}f''(0)$$

Further:

$$f'''(x) = 3(2)b_3 + 4(3)(2)b_4x + \cdots$$

so

$$f'''(0) = 3(2)b_3 \qquad \text{implying} \qquad b_3 = \frac{1}{3(2)}f'''(0)$$

Continuing in this way we easily find that (remembering that $0! = 1$)

$$b_n = \frac{1}{n!}f^{(n)}(0) \qquad n = 0, 1, 2, \ldots$$

where $f^{(n)}(0)$ means the value of the n^{th} derivative at $x = 0$ and $f^{(0)}(0)$ means $f(0)$.
Bringing all these results together we find

Key Point

If $f(x)$ can be differentiated as often as we please:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots = \sum_{p=0}^{\infty} \frac{1}{p!}f^{(p)}(0)x^p$$

This is called the **Maclaurin expansion** of $f(x)$

Example Find the Maclaurin expansion of $\cos x$.

Solution

Here $f(x) = \cos x$ and, differentiating a number of times:

$$f(x) = \cos x, \qquad f'(x) = -\sin x, \qquad f''(x) = -\cos x, \qquad f'''(x) = \sin x \quad \text{etc.}$$

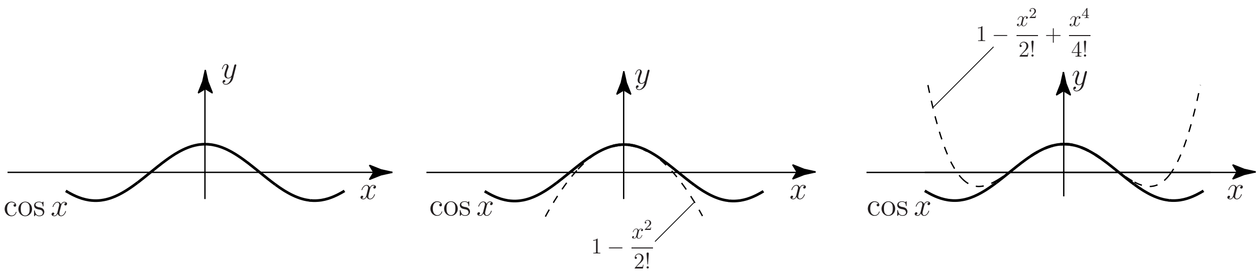
Thus, evaluating each of these at $x = 0$:

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0 \text{ etc.}$$

Now, substituting into $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$, implies

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

The reader should confirm (by finding the radius of convergence) that this series is convergent for **all** values of x. The geometrical approximation to $\cos x$ by the first few terms of its Maclaurin series are shown in the following diagram.



Find the Maclaurin expansion of $\ln(1 + x)$. (Note that we **cannot** find a Maclaurin expansion of the function $\ln x$ since this function cannot be differentiated at $x = 0$).

Find the first few derivatives of $f(x) = \ln(1 + x)$

Your solution

$f'(x) =$ $\qquad$ $f''(x) =$ $\qquad$ $f'''(x) =$ $\qquad$ $f''''(x) =$

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = \frac{-1}{(1+x)^2}, \quad f'''(x) = \frac{2}{(1+x)^3}, \quad f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n}$$

Now obtain $f(0),\ f'(0),\ f''(0),\ f'''(0), \ldots$

Your solution

$f(0) =$ $\quad f'(0) =$ $\quad f''(0) =$ $\quad f'''(0) =$

$f(0) = 0 \qquad f'(0) = 1, \qquad f''(0) = -1, \qquad f'''(0) = 2, \qquad f^{(n)}(0) = (-1)^{n+1}(n-1)!$

Hence, obtain the Maclaurin expansion of $\ln(1+x)$.

Your solution

$\ln(1+x) =$

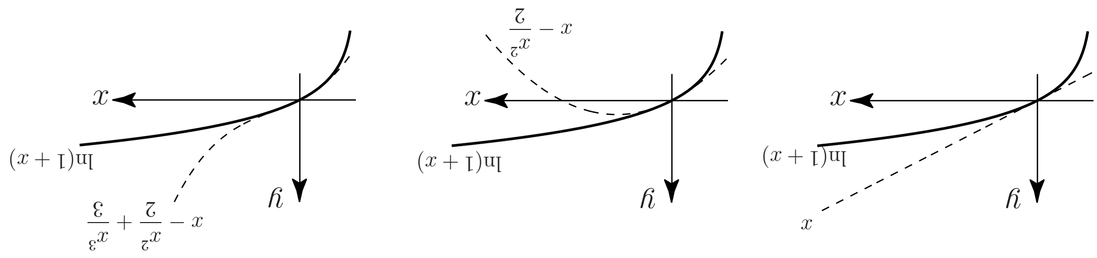
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots + \frac{(-1)^{n+1}}{n}x^n + \cdots$ This has already been obtained in Section 16.4.

Now obtain the interval of convergence?

Your solution

Radius of convergence $\quad R =$

$R = 1$. Also at $x = 1$ the series is convergent (alternating harmonic series) and at $x = -1$ the series is divergent. Hence this Maclaurin expansion is only valid if $-1 < x \leq 1$. The geometrical closeness of the polynomial terms with the function $\ln(1+x)$ for $-1 < x \leq 1$ are displayed in the following diagram.



Note that when $x = 1$

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$$

so the alternating harmonic series converges to $\ln 2 \simeq 0.693$, a claim first made in Section 16.2.

Example Find the Maclaurin expansion of $e^x \ln(1+x)$.

Solution

Here, instead of finding the derivatives of $f(x) = e^x \ln(1+x)$, we can multiply together the Maclaurin expansions we already know:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{all } x$$

and

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots \qquad -1 < x \leq 1$$

The resulting power series will only be convergent if $-1 < x \leq 1$. That is

$$\begin{aligned} e^x \ln(1+x) &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\ &\qquad + x^2 - \frac{x^3}{2} + \frac{x^4}{3} + \cdots \\ &\qquad\qquad + \frac{x^3}{2} - \frac{x^4}{4} \cdots \\ &\qquad\qquad\qquad + \frac{x^4}{6} \cdots \\ &= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^5}{40} + \cdots \qquad -1 < x \leq 1 \end{aligned}$$

(You must take care not to miss relevant terms when carrying through the multiplication).

The Maclaurin expansion of a product of two functions: $f(x)g(x)$ is obtained by multiplying together the Maclaurin expansions of $f(x)$ and of $g(x)$ and collecting like terms together. The product series will have a radius of convergence equal to the **smaller** of the two separate radii of convergence.

Find the Maclaurin expansion of $\cos^2 x$ up to powers of x^4. Hence write down the expansion of $\sin^2 x$ to powers of x^6.

First, write down the expansion of $\cos x$

Your solution

$\cos x =$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

Now, by multiplication, find the expansion of $\cos^2 x$. (The reader could try to obtain the power series expansion for $\cos^2 x$ by using the trigonometric identity $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$).

Your solution

$\cos^2 x =$

$$\cos^2 x = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\right)$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\right) + \left(-\frac{x^2}{2!} + \frac{x^4}{4} \cdots\right) + \left(\frac{x^4}{4!} \cdots\right) + \cdots = 1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} \cdots$$

Now obtain the expansion of $\sin^2 x$.

Your solution

$\sin^2 x =$

$$\sin^2 x = 1 - \cos^2 x = 1 - \left(1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} + \cdots\right) = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} + \cdots$$

3. Differentiation of Maclaurin Series

We have already noted that, by the binomial series,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad |x| < 1$$

Thus, with x replaced by $-x$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \qquad |x| < 1$$

Also, we have obtained the Maclaurin expansion of $\ln(1+x)$:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad -1 < x \leq 1$$

Now, we differentiate both sides with respect to x:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

This demonstrates that the Maclaurin expansion of a function $f(x)$ may be differentiated term by term to give a series which will be the Maclaurin expansion of $\frac{df}{dx}$.
As we noted in Section 16.4 the derived series will have the **same** radius of convergence as the original series.

Find the Maclaurin expansion of $(1-x)^{-3}$.

First write down the expansion of $(1-x)^{-1}$

Your solution

$$\frac{1}{1-x}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \qquad |x| < 1$$

Now, by differentiation, obtain the expansion of $\frac{1}{(1-x)^2}$

Your solution

$$\frac{1}{(1-x)^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-x}\right) =$$

$$\frac{1}{(1-x)^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(1 + x + x^2 + \cdots\right) = 1 + 2x + 3x^2 + 4x^3$$

Differentiate again to obtain the expansion of $(1-x)^{-3}$.

Your solution

$$\frac{1}{(1-x)^3} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{(1-x)^2}\right) = \frac{1}{2}\left[\qquad\qquad\right]$$

$$\frac{1}{(1-x)^3} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{(1-x)^2}\right) = \frac{1}{2}\left[2 + 6x + 12x^2 + \cdots\right] = 1 + 3x + 6x^2 + 10x^3 + \cdots$$

The final series: $1 + 3x + 6x^2 + 10x^3 + \cdots$ has radius of convergence $R = 1$ since the original series, before differentiation, has this radius of convergence (but this can also be found directly using the formula $R = \lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right|$ and using the fact that the coefficient of the n^{th} term is $b_n = \frac{1}{2}n(n+1)$).

4. The Taylor Series

The **Taylor series** is a generalisation of the Maclaurin series being a power series developed in powers of $(x - x_0)$ rather than in powers of x. Thus

Key Point

If the function $f(x)$ can be differentiated as often as we please at $x = x_0$ then:

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \cdots$$

This is called the Taylor series of $f(x)$ about the point x_0.

The reader will see that the Maclaurin expansion is obtained if x_0 is chosen to be zero.

Obtain the Taylor series expansion of $\frac{1}{1-x}$ about $x = 2$. (That is, find a power series in powers of $(x - 2)$).

First, obtain the derivatives of $f(x) = \frac{1}{1-x}$

Your solution

$f'(x) =$, $f''(x) =$, $f'''(x) =$

$f'(x) = \frac{1}{(1-x)^2}$, $\quad f''(x) = \frac{2}{(1-x)^3}$, $\quad \cdots \quad f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$

Now evaluate these derivatives at $x_0 = 2$.

Your solution

$f'(2) =$ $\quad f''(2) =$ $\quad \cdots$

$f'(2) = 1$, $\quad f''(2) = -2, \ldots$, $\quad f^{(n)}(2) = (-1)^{n+1}n!$

Hence, write down the Taylor expansion of $f(x) = \frac{1}{1-x}$ about $x = 2$

Your solution

$\frac{1}{1-x} =$

$\frac{1}{1-x} = -1 + (x-2) - (x-2)^2 + (x-2)^3 + \cdots + (-1)^{n+1}(x-2)^n + \cdots$

The reader should confirm that this series is convergent if $|x-2| < 1$. In the diagram following some of the terms from the Taylor series are plotted to compare with $\frac{1}{(1-x)}$.