Contents 17

conics and polar coordinates

Learning outcomes

In this workbook you will learn about some of the most important curves in the whole of mathematics - the conic sections: the ellipse, the parabola and the hyperbola. You will learn how to recognise these curves and how to describe them in Cartesian and in polar form. In the final block you will learn how to describe cruves using a parametric approach and, in particular, how the conic sections are described in parametric form.

Time allocation

You are expected to spend approximately five hours of independent study on the material presented in this workbook. However, depending upon your ability to concentrate and on your previous experience with certain mathematical topics this time may vary considerably.

Conic Sections

17.1

Introduction

The *conic sections* (or conics) - the ellipse, the parabola and the hyperbola - play an important role both in mathematics and in the application of mathematics to engineering. In this section we look in detail at the equations of the conics in both standard form and general form.

Although there are various ways that can be used to define a conic we concentrate, in this section on defining conics using Cartesian coordinates (x, y). However, at the end of this section we examine an alternative way to obtain the conics.

Prerequisites

Before starting this Section you should . . .

① thoroughly understand the various techniques of differentiation

Learning Outcomes

After completing this Section you should be able to . . .

✓ appreciate how conics are obtained as curves of intersection of a double-cone with a plane

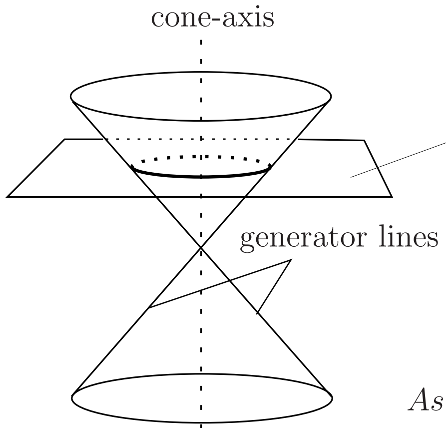
✓ state the standard form of the equations of ellipse, the parabola and hyperbola

✓ classify quadratic expressions in x, y in terms of conics

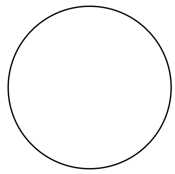
1. The Ellipse, Parabola and Hyperbola

Mathematicians, engineers and scientists encounter numerous functions in their work: polynomials, trigonometric and hyperbolic functions amongst them. However, throughout the history of science one group of functions, the conics, arise time and time again not only in the development of mathematical theory but also in practical applications. The conics were first studied by the Greek mathematician Apollonius more than 200 years BC.

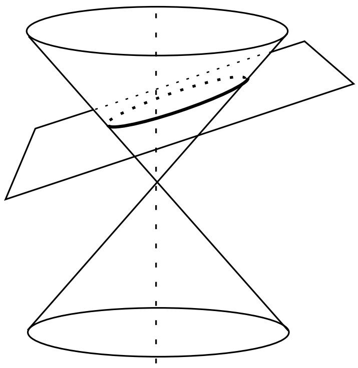
Essentially, the conics form that class of curves which are obtained when a double cone is intersected by a plane. There are three main types: the **ellipse**, the **parabola** and the **hyperbola**. From the ellipse we obtain the **circle** as a special case, and from the hyperbola we obtain the **rectangular hyperbola** as a special case. These curves are illustrated in the following figures.



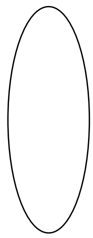
Circle: obtained by intersection of a plane perpendicular to the cone-axis with cone

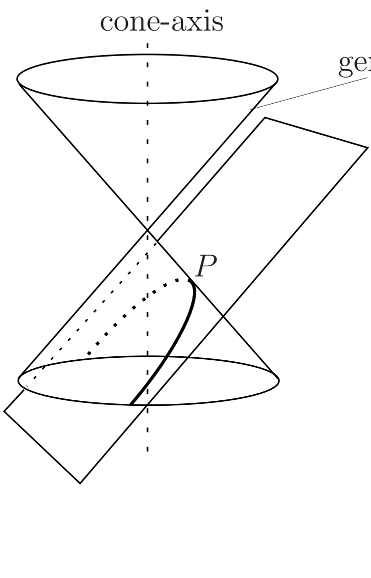


As the plane of intersection rotates the other conics are obtained

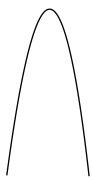


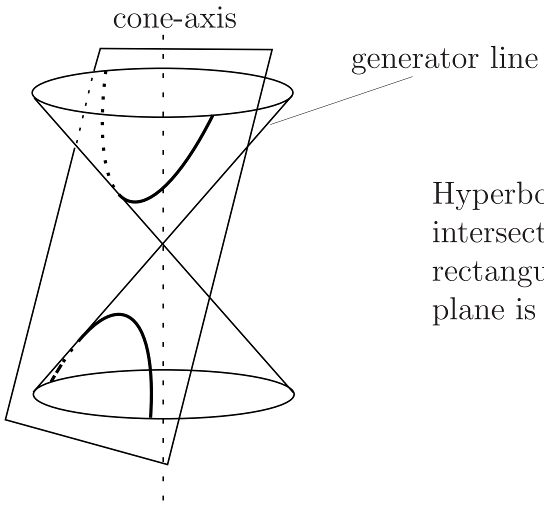
Ellipse: obtained by a plane, which is not perpendicular to the cone-axis, but cutting the cone in a closed curve.
Various ellipses are obtained as the plane continues to rotate.



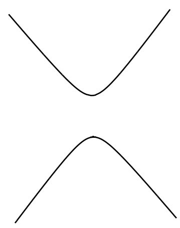


Parabola: obtained when the plane is parallel to the generator of the cone.
Different parabolas are obtained as the point P moves along a generator.





Hyperbola: obtained when the plane intersects both parts of the cone. The rectangular hyperbola is obtained when the plane is parallel to the cone-axis.



The Ellipse

We are all aware that the paths followed by the planets around the sun are elliptical. However, more generally the ellipse occurs in many areas of engineering. The standard form of an ellipse is shown in Figure 1.

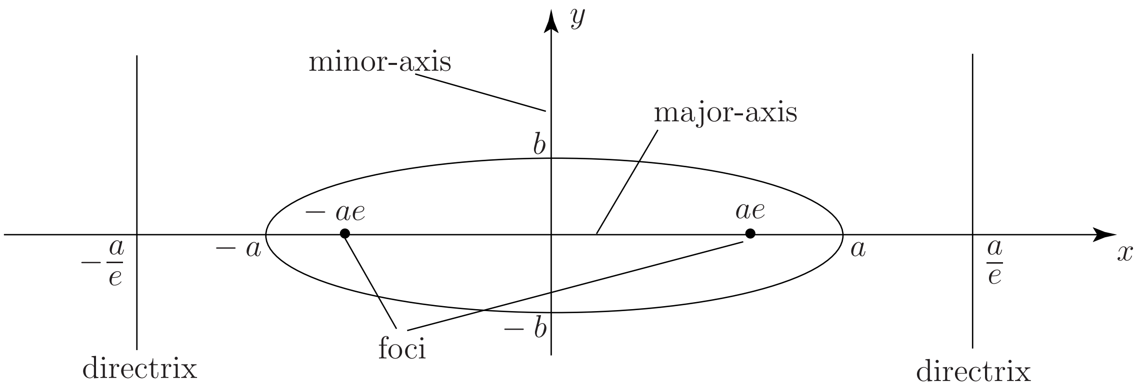


Figure 1.

If $a > b$ (as we have drawn the figure) then the $x-$axis is called the *major-axis* and the $y-$axis is called the *minor-axis*. On the other hand if $b > a$ then the $y-$axis is called the major-axis and the $x-$axis is then the minor-axis. Two points, inside the ellipse are of importance; these are the **foci**. If $a > b$ these are located at coordinate positions $\pm ae$ (or at $\pm be$ if $b > a$) on the major-axis in which e, called the **eccentricity**, is given by

$$e^2 = 1 - \frac{b^2}{a^2} \qquad (b < a) \qquad \text{or by} \qquad e^2 = 1 - \frac{a^2}{b^2} \qquad (a < b)$$

The foci of an ellipse have the property that if light rays are emitted from one focus then on reflection at the elliptic curve they pass through at the other focus.

Key Point

The standard Cartesian equation of the ellipse with its centre at the origin is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This ellipse has intercepts on the x-axis at $x = \pm a$ and on the y-axis at $\pm b$. The curve is also symmetrical about both axes. The curve reduces to a circle in the special case in which $a = b$.

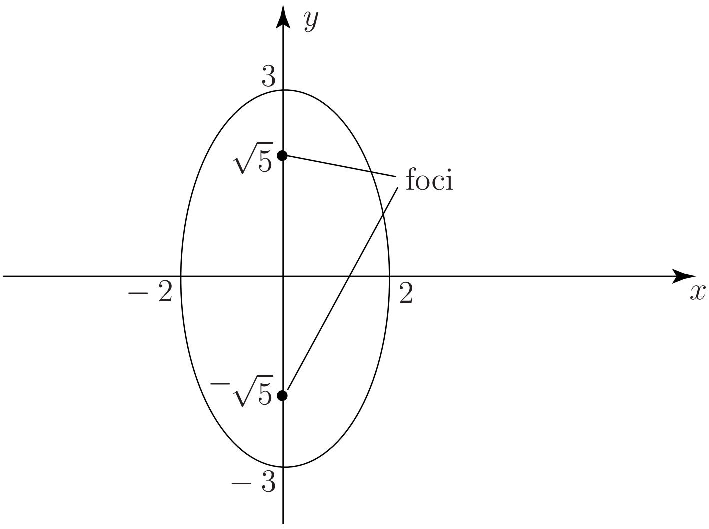
Example (a) Sketch the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ (b) Find the eccentricity e (c) Locate the positions of the foci.

Solution

(a) We can calculate the values of y as x changes from 0 to 2

x	0	0.30	0.60	0.90	1.20	1.50	1.80	2
y	3	2.97	2.86	2.68	2.40	1.98	1.31	0

From this table of values, and using the symmetry of the curve, a sketch can be drawn.



Here $b = 3$ and $a = 2$ so that the $y-$axis is the major axis and the $x-$axis is the minor axis.

(b)

$$e^2 = 1 - \frac{a^2}{b^2} = 1 - \frac{4}{9} = \frac{5}{9} \qquad \therefore \quad e = \frac{\sqrt{5}}{3}$$

(c) The foci are located at $\pm\sqrt{5}$ on the $y-$axis.

The Key Point above gives the equation of the ellipse with its centre at the origin. If the centre of the ellipse has coordinates $(\alpha,\ \beta)$ and still has its axes parallel to the x and $y-$axes the standard equation becomes

$$\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1.$$

The Circle

The circle is a special case of the ellipse; it occurs when $a = b = r$ so the equation becomes

$$\frac{x^2}{r^2}+\frac{y^2}{r^2}=1 \qquad \text{or, more commonly} \quad x^2+y^2=r^2$$

Here, the *centre* of the circle is located at the origin $(0, 0)$ and the *radius* of the circle is r. If the centre of the circle at a point (α, β) then we use the form:

$$(x-\alpha)^2+(y-\beta)^2=r^2$$

Key Point

The equation of a circle with centre at (α, β) and radius r is

$$(x-\alpha)^2+(y-\beta)^2=r^2$$

In Figure 2 we have drawn a number of circles
The reader should confirm that the equations of these circles are

circle	equation
A	$(x-1)^2+(y-1)^2=1$
B	$(x-3)^2+(y-1)^2=1$
C	$(x+0.5)^2+(y+2)^2=1$
D	$(x-2)^2+(y+2)^2=0.25$
E	$(x+0.5)^2+(y-2.5)^2=1$

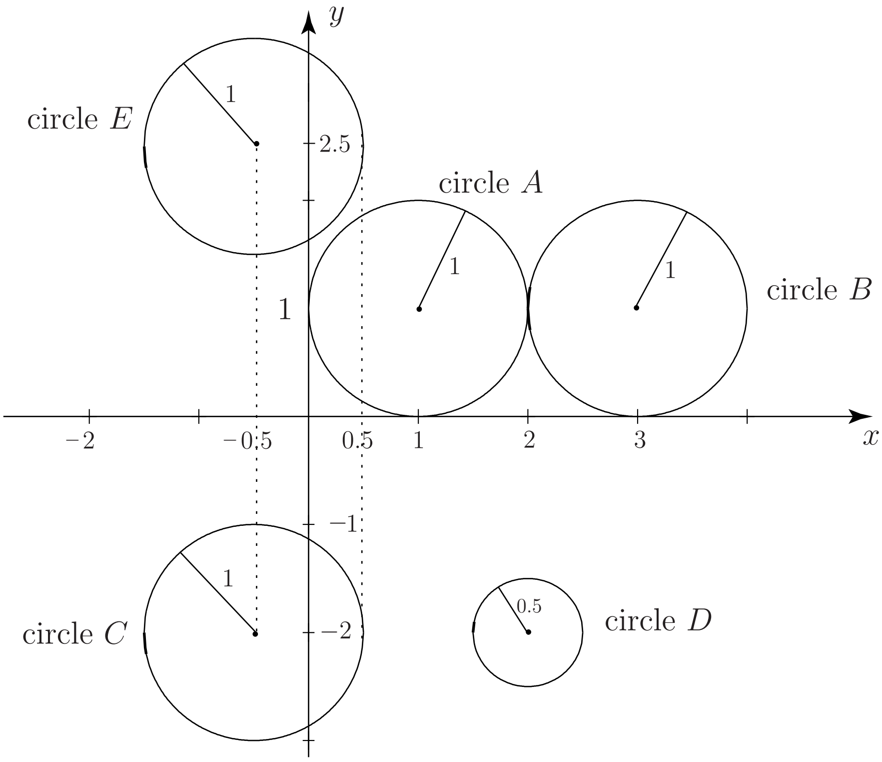


Figure 2.

Example Show that the expression

$$x^2 + y^2 - 2x + 6y + 6 = 0$$

represents the equation of a circle. Find its centre and radius.

Solution

We shall see later how to recognise this as the equation of a circle simply by examination of the coefficients of quadratic terms x^2, y^2. However, in the present example we will use the process of completing the square, for x and for y, to show that the expression can be written in standard form. Now $x^2 + y^2 - 2x + 6y + 6 \equiv x^2 - 2x + y^2 + 6y + 6$. Also,

$$x^2 - 2x \equiv (x-1)^2 - 1 \quad \text{and} \quad y^2 + 6y \equiv (y+3)^2 - 9.$$

Hence we can write

$$x^2 + y^2 - 2x + 6y + 6 \equiv (x-1)^2 - 1 + (y+3)^2 - 9 + 6 = 0$$

or, taking the free constants to the right-hand side:

$$(x-1)^2 + (y+3)^2 = 4.$$

By comparing this with the standard form we conclude this represents the equation of a circle with centre at coordinate position $(1, -3)$ and radius 2.

Find the centre and radius of each of the following circles:

(a) $x^2 + y^2 - 4x - 6y = -12$ (b) $2x^2 + 2y^2 + 4x + 1 = 0$

Your solution

(a) centre: $(2, 3)$ radius 1 (b) centre: $(-1, 0)$ radius $\sqrt{2}/2$.

The Parabola

The standard form of the parabola is shown in Figure 3.
Here the $x-$axis is the line of symmetry of the parabola.

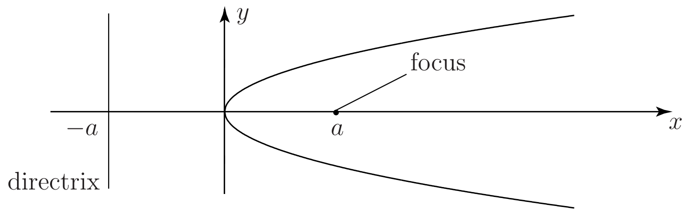


Figure 3.

Key Point

The standard equation of the parabola is

$$y^2 = 4ax$$

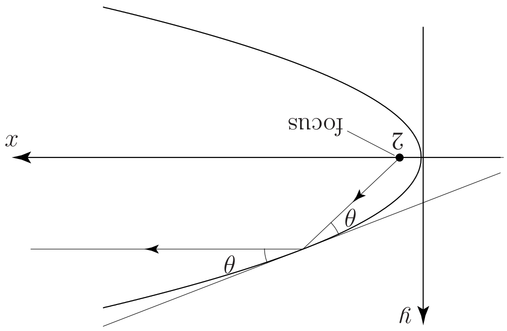
with focus at $(a, 0)$.

It can be shown that light rays parallel to the $x-$axis will, on reflection from the parabolic curve, come together at the focus. This is an important property and is used in the construction of some kinds of telescopes, of satellite dishes and of car headlights.

Sketch the curve $y^2 = 8x$. Find the position of the focus and confirm its light-focusing property.

Your solution

This is a standard parabola ($y^2 = 4ax$) with $a = 2$. Thus the focus is located at coordinate position (2,0).



If your sketch is sufficiently accurate you should find that light-rays (lines) parallel to the $x-$axis when reflected off the parabolic surface pass through the focus. (Draw a tangent at the point of reflection and ensure that the angle of incidence (θ say) is the same as the angle of reflection.)

By changing the equation of the parabola slightly we can change the position of the parabola along the $x-$axis. See Figure 4.

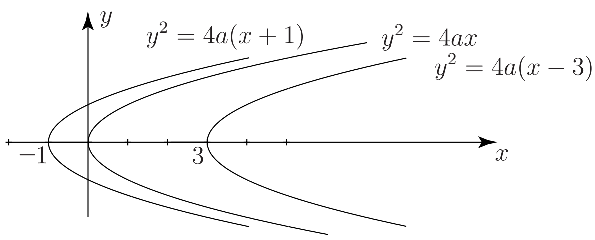


Figure 4. parabola $y = 4a(x - b)$ has its vertex at $x = b$.

We can also have parabolae where the $y-$axis is the line of symmetry (see Figure 5). In this case the standard equation would be

$$x^2 = 4ay \qquad \text{or} \qquad y = \frac{x^2}{4a}$$

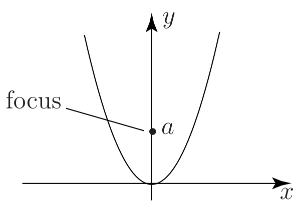
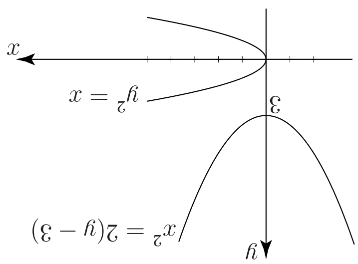


Figure 5.

Sketch the curves $y^2 = x$ and $x^2 = 2(y-3)$

Your solution



The focus of the parabola $y^2 = 4a(x-b)$ is located at coordinate position $(a+b,\ 0)$. Also, changing the value of a changes the convexity of the parabola (see Figure 6).

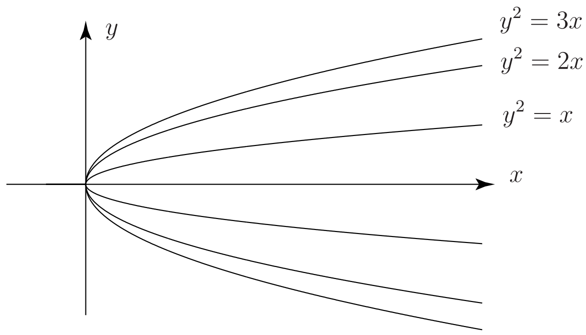


Figure 6.

The Hyperbola

The standard form of the hyperbola is shown in Figure 7(a).
This has standard equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

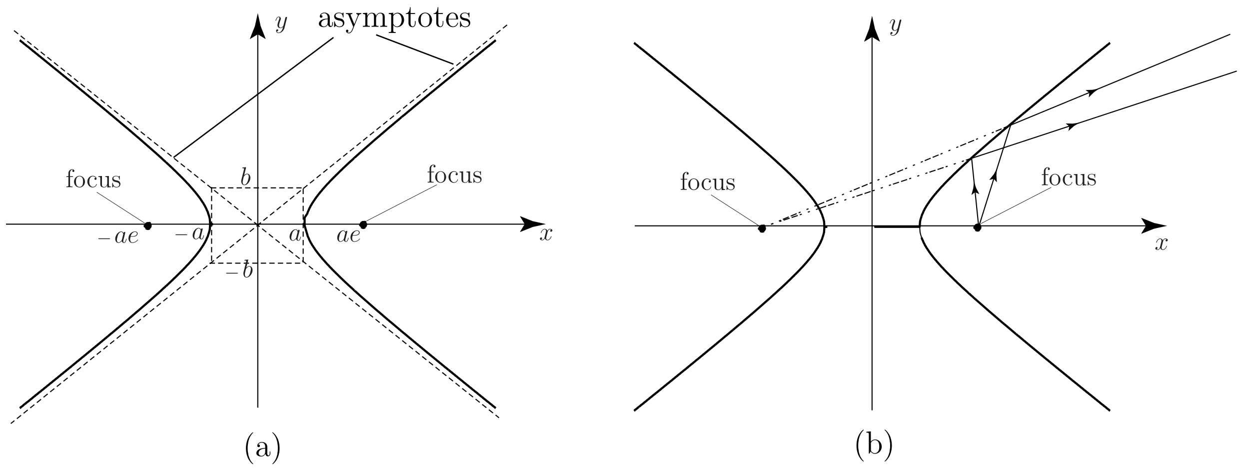


Figure 7.

The eccentricity, e, is defined by

$$e^2 = 1 + \frac{b^2}{a^2} \qquad (e > 1)$$

Note the change in sign when compared to the equivalent expressions for the ellipse. The lines $y = \pm\frac{b}{a}x$ are asymptotes to the hyperbola (these are the lines to which each branch of the hyperbola approach as $x \to \pm\infty$).
If light is emitted from one focus then on hitting the hyperbolic curve it is reflected in such a way as to appear to be coming from the *other* focus. See Figure 7(b). The hyperbola has fewer uses in applications than the other conic sections and so we will not dwell here on its properties.

General Conics

The conics we have considered above - the ellipse, the parabola and the hyperbola - have all been presented in standard form:- their axes are parallel to either the x or $y-$axes. However, conics may be rotated to any angle with respect to the axes: they clearly remain conics, but what equations do they have?
It can be shown that the equation of *any* conic, can be described by the quadratic expression

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where A, B, C, D, E, F are constants.
If not all of A, B, C are zero the graph of this equation is

(i) an ellipse if $B^2 < 4AC$ (circle if $A = C$ **and** $B = 0$)

(ii) a parabola if $B^2 = 4AC$

(iii) a hyperbola if $B^2 > 4AC$

Example Classify each of the following expressions as ellipse, parabola or hyperbola;

(a) $x^2 + 2xy + 3y^2 + x - 1 = 0$

(b) $x^2 + 2xy + y^2 - 3y + 7 = 0$

(c) $2x^2 + xy + 2y^2 - 2x + 3y = 6$

Solution

(a) Here $A = 1,\ B = 2,\ C = 3 \quad \therefore \quad B^2 < 4AC$. This is an ellipse

(b) Here $A = 1,\ B = 2,\ C = 1 \quad \therefore \quad B^2 = 4AC$. This is a parabola

(c) Here $A = 2,\ B = 1,\ C = 2 \quad \therefore \quad B^2 < 4AC$ also $A = C$. This is an ellipse.

Classify each of the following conics:

(a) $x^2 - 2xy - 3y^2 + x - 1 = 0$

(b) $2x^2 + xy - y^2 - 2x + 3y = 0$

(c) $4x^2 - y + 3 = 0$

(d) $-x^2 - xy - y^2 + 3x = 0$

Your solution

(a) $A = 1, B = -2, C = -3 \quad B^2 > 4AC \quad \therefore$ hyperbola

(b) $A = 2, B = 1, C = -1 \quad B^2 > 4AC \quad \therefore$ hyperbola

(c) $A = 4, B = 0, C = 0 \quad B^2 = 4AC \quad \therefore$ parabola

(d) $A = -1, B = -1, C = -1 \quad B^2 < 4AC,\ A = C \quad \therefore$ ellipse

Exercises

1. The equation $9x^2 + 4y^2 - 36x + 24y - 1 = 0$ represents an ellipse. Find its centre, the semi-major and semi-minor axes and the coordinate positions of the foci.
2. Find the equation of a circle of radius 3 which has its centre at $(-1, 2.2)$
3. Find the centre and radius of the circle $x^2 + y^2 - 2x - 2y - 5 = 0$
4. Find the position of the focus of the parabola $y^2 - x + 3 = 0$
5. Classify each of the following conics
   (a) $x^2 + 2x - y - 3 = 0$
   (b) $8x^2 + 12xy + 17y^2 - 20 = 0$
   (c) $x^2 + xy - 1 = 0$
   (d) $4x^2 - y^2 - 4y = 0$
   (e) $6x^2 + 9y^2 - 24x - 54y + 51 = 0$
6. An asteroid has an elliptical orbit around the Sun. The major axis is of length 5×10^8 km. If the distance between the foci is 4×10^8 km find the equation of the orbit.

Answers

1. centre: $(2, -3)$, semi-major 3, semi-minor 2, foci: $(2, -3 \pm \sqrt{5})$
2. $(x + 1)^2 + (y - 2.2)^2 = 9$
3. centre: $(1, 1)$ radius $\sqrt{7}$
4. $y^2 = (x - 3) \quad \therefore \quad a = 1, \quad b = -3$. Hence focus is at coordinate position $(4, 0)$.
5. (a) parabola with vertex $(-1, -4)$
   (b) ellipse
   (c) hyperbola
   (d) hyperbola
   (e) ellipse with centre $(2,3)$
6. $9x^2 + 25y^2 = 5.625 \times 10^7$