Parametric Curves

17.3

Introduction

In this section we examine yet another way of defining curves - the parametric description. We shall see that this is, in some ways, far more useful than either the Cartesian description or the polar form. Although we shall only study planar curves (curves lying in a plane) the parametric description can be easily generalised to the description of spatial curves which twist and turn in three dimensional space.

Prerequisites

Before starting this Section you should . . .

① be familiar with Cartesian coordinates

② be familiar with trigonometric functions and how to manipulate then

③ be able to differentiate simple functions

④ be able to locate turning points and distinguish between maxima and minima.

Learning Outcomes

After completing this Section you should be able to . . .

✓ sketch planar curves given in parametric form

✓ understand that the same curve can be described using many different parametrisations

✓ recognise some conics given in parametric form

1. Parametric Curves

In this section we explore the use of a parameter t in the description of curves. We shall see that it has some advantages over the more usual Cartesian description. We start with a simple example.

Example Plot the curve

$$\underbrace{x = 2\cos t \qquad y = 3\sin t}_{\text{parametric equations of the curve}} \qquad\qquad \underbrace{0 \leq t \leq \frac{\pi}{2}}_{\text{parameter range}}$$

Solution

The approach to sketching the curve is straightforward. We simply give the parameter t various values as it ranges through $0 \to \frac{\pi}{2}$ and, for each value of t, calculate corresponding values of (x, y) which are then plotted on a Cartesian xy plane. The value of t and the corresponding values of x, y are recorded in the following table:

t	0	$\frac{\pi}{20}$	$\frac{2\pi}{20}$	$\frac{3\pi}{20}$	$\frac{4\pi}{20}$	$\frac{5\pi}{20}$	$\frac{6\pi}{20}$	$\frac{7\pi}{20}$	$\frac{8\pi}{20}$	$\frac{9\pi}{20}$	$\frac{10\pi}{20}$
x	2	1.98	1.90	1.78	1.62	1.41	1.18	0.91	0.62	0.31	0
y	0	0.47	0.93	1.36	1.76	2.12	2.43	2.67	2.85	2.96	3

Plotting the (x, y) coordinates gives the curve in Figure 1.

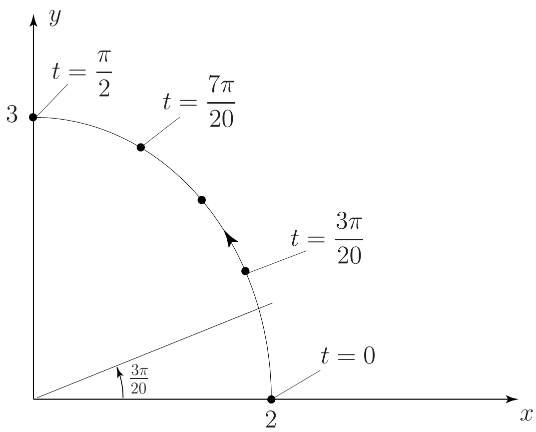


Figure 1.

The curve resembles part of an ellipse. This can be verified by eliminating t from the parametric equations to obtain an expression involving x, y only. If we divide the first parametric equation by 2 and the second by 3, square both and add we obtain

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \cos^2 t + \sin^2 t \equiv 1$$

i.e.

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

which we easily recognise as an ellipse whose major-axis is the $y-$axis. Also, as t ranges from $0 \to \frac{\pi}{2}$ it is clear from $x = 2\cos t$ that x decreases from $2 \to 0$ and, from $y = 3\sin t$, that y increases from $0 \to 3$. We conclude that the parametric equations $x = 2\cos t,\ y = 3\sin t$ together with the parametric range $0 \leq t \leq \pi/2$ describe that part of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ in the positive quadrant. On the curve in Figure 1 we have used an arrow to indicate the direction that we move along the curve as t **increases** from its initial value 0.

Plot the curve

$$x = t + 1 \qquad y = 2t^2 - 3 \qquad 0 \leq t \leq 1$$

Do you recognise this curve as a conic section?

First construct a table of (x, y) values as t ranges from $0 \to +1$

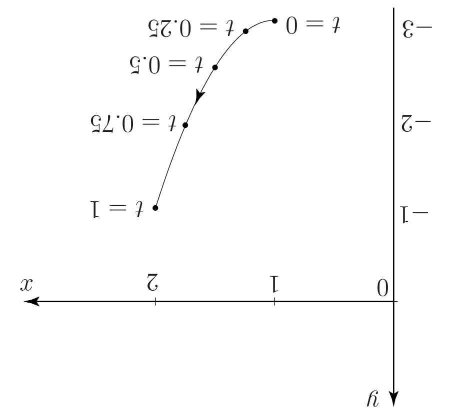
Your solution

t	0	0.25	0.5	0.75	1
x					
y					

t	0	0.25	0.5	0.75	1
x	1	1.25	1.5	1.75	2
y	-3	-2.88	-2.5	-1.88	-1

Now plot the points on a Cartesian plane

Your solution



Now eliminate the $t-$variable from $x = t + 1$, $y = 2t^2 - 3$ to obtain the xy form of the curve.

Your solution

$y = 2x^2 - 4x - 1$ which is the equation of a parabola.

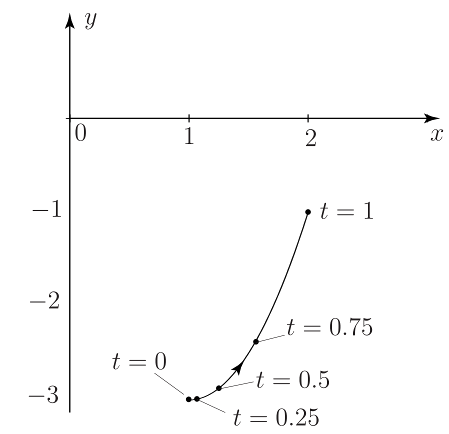
Example Sketch the curve

$$x = t^2 + 1 \qquad y = 2t^4 - 3 \qquad 0 \leq t \leq 1$$

Solution

This is very similar to the previous guided exercise (except for t^4 replacing t^2 in the expression for y and t^2 replacing t in the expression for x. The corresponding table of values is

t	0	0.25	0.5	0.75	1
x	1	1.06	1.25	1.56	2
y	-3	-2.99	-2.88	-2.37	-1



We see that this is **identical** to the curve drawn previously. This is confirmed by eliminating the $t-$parameter from the expressions defining x, y. Here $t^2 = x - 1$ so $y = 2(x-1)^2 - 3$ which is the same as obtained in the guided exercise. The main difference is that particular values of t locate (in general) different (x, y) points on the curve.

We conclude that a given curve in the xy plane can have many (in fact infinitely many) parametric descriptions.

Show that the two parametric descriptions

(a) $x = \cos t \qquad y = \sin t \qquad 0 \le t \le \frac{\pi}{2}$

(b) $x = t \qquad y = \sqrt{1-t^2} \qquad 0 \le t \le 1$

describe the same curve.

Your solution

Eliminate t from the equations in (a)

$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$

Your solution

Eliminate t from the equations in (b)

$y = \sqrt{1 - x^2} \quad \therefore \quad y^2 = 1 - x^2 \quad \text{or} \quad x^2 + y^2 = 1$

i.e. both parametric parametric descriptions represent (part of) the circle centred at the origin of radius 1.

2. General Parametric Form

We will assume that any curve in the xy plane may be written in parametric form:

$$\underbrace{x = g(t) \qquad y = h(t)}_{\text{parametric equations of the curve}} \qquad\qquad \underbrace{t_0 \leq t \leq t_1}_{\text{parameter range}}$$

in which $g(t),\ h(t)$ are **given** functions of t and the parameter t ranges over the values $t_0 \to t_1$. As we give values to t within this range then corresponding values of x, y are calculated from $x = g(t)$, $y = h(t)$ which can then be plotted on an xy plane.

In Workbook 12, section 3, we discovered how to obtain the derivative $\frac{dy}{dx}$ from a knowledge of the parametric derivatives $\frac{dy}{dt}$ and $\frac{dx}{dt}$. We found

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{\left(\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}\right)}{\left(\frac{dx}{dt}\right)^3}$$

Note that derivatives with respect to the parameter t are often denoted by a dot

$$\frac{dx}{dt} \equiv \dot{x} \qquad \frac{dy}{dt} \equiv \dot{y} \qquad \frac{d^2x}{dt^2} \equiv \ddot{x} \quad \text{etc}$$

so that

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$$

Knowledge of the derivative is sometimes useful in curve sketching.

Example Sketch the curve

$$x = t^3 + 3t^2 + 2t \qquad y = 3 - 2t - t^2 \qquad -3 \leq t \leq 1.$$

Solution

$$x = t^3 + 3t^2 + 2t = t(t^2 + 3t + 2) = t(t + 2)(t + 1)$$
$$y = 3 - 2t - t^2 = -(t^2 + 2t - 3) = -(t + 3)(t - 1)$$

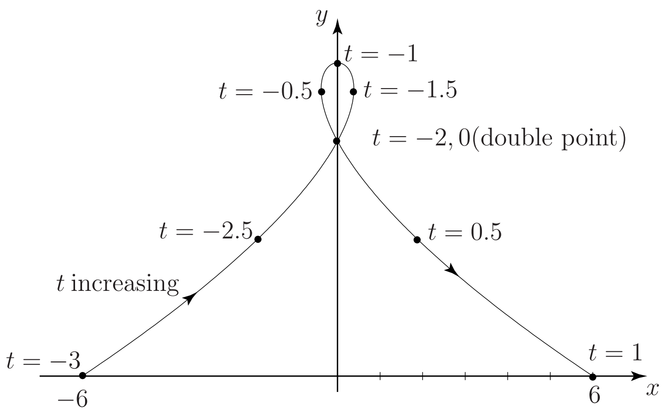
so that x vanishes when $t = 0,\ -1,\ -2$ and y vanishes when $t = -3,\ 1$. We easily calculate the values of x, y at these and at other values of t.

t	-3	-2.50	-2	-1.50	-1	-0.50	0	0.50
x	-6	-1.88	0	0.38	0	-0.38	0	1.88
y	0	1.75	3	3.75	4	3.75	3	1.75

We see $t = -2$ and $t = 0$ give rise to the **same** coordinate values for (x, y). This represents a **double-point** in the curve which is one where the curve crosses itself. Now

$$\frac{dx}{dt} = 3t^2 + 6t + 2, \quad \frac{dy}{dt} = -2 - 2t \qquad \therefore \qquad \frac{dy}{dx} = \frac{-2(1 + t)}{3t^2 + 6t + 2}$$

so there is a turning point when $t = -1$. The reader is urged to calculate $\frac{d^2y}{dx^2}$ and to show that this is negative when $t = -1$ (i.e. $x = 0,\ y = 4$) indicating a maximum when. (The reader should check that vertical tangents occur at $t = -0.43,\ -1.47$ (to 2 d.p.)). We can now make a reasonable sketch of the curve;



3. Standard Forms of Conic Sections in Parametric Form

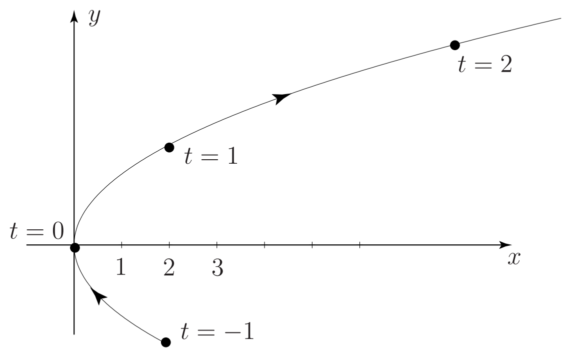
We have seen above that, given a curve in the xy plane, there is no unique way of representing it in parametric form. However, for some commonly occurring curves, particularly the conics, there are accepted standard parametric equations.

The Parabola

The standard parametric equations for a parabola are:

$$x = at^2 \qquad y = 2at$$

(the range for the parameter t is usually omitted). Clearly, we have $t = \frac{y}{2a}$ and by eliminating t $x = a\left(\frac{y^2}{4a^2}\right)$ or $y^2 = 4ax$ which we recognise as the standard Cartesian description of a parabola. (As an illustration, the sketch shows the curve with $a = 2$ and $-1 \leq t \leq 2.3$)



The Ellipse

Here, the standard equations are

$$x = a\cos t \qquad\qquad y = b\sin t$$

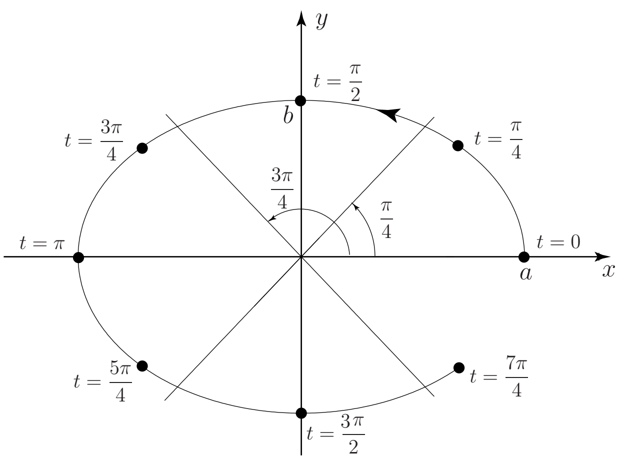
Again, eliminating t (dividing the first equation by a, the second by b, squaring and adding) we have

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \cos^2 t + \sin^2 t \equiv 1$$

or, in more familiar form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

If we choose the range for the parameter t as $0 \leq t \leq \frac{7\pi}{4}$ the following segment of the ellipse is obtained.



Here we note that (except in the special case when $a = b$, giving a circle) the parameter t is **not** the angle that the radial line makes with the the positive $x-$axis.

The Hyperbola

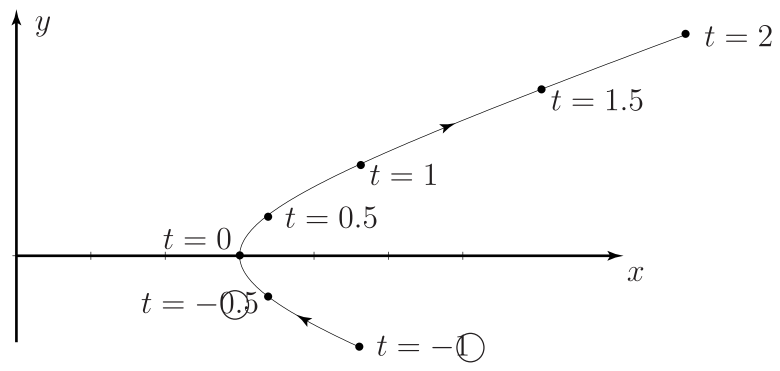
The standard equations are

$$x = a\cosh t \qquad y = b\sinh t$$

In this case, to eliminate t we use the identity $\cosh^2 t - \sinh^2 t = 1$ giving rise to the equation of the hyperbola in Cartesian form:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

In the following curve we have chosen a parameter range $-1 \leq t \leq 2$.



To obtain the complete curve the parameter range $-\infty < t < \infty$ must be used. These parametric equations only give the right-hand branch of the hyperbola. To obtain the left-hand branch we would use

$$x = -a\cosh t \qquad y = b\sinh t$$

Exercises

1. In the followng examples sketch the given parametric curves, Also, eliminate the parameter to give the Cartesian equation in x and y.

   (a) $x = t, \quad y = 2 - t \qquad 0 \leq t \leq 1$

   (b) $x = 2 - t, \quad y = t + 1 \qquad 0 \leq t \leq \infty$

   (c) $x = \frac{2}{t} \quad y = t - 2 \qquad 0 < t < 3$

   (d) $x = 3\sin\frac{\pi t}{2} \quad y = 4\cos\frac{\pi t}{2} \qquad -1 \leq t \leq 0.5$

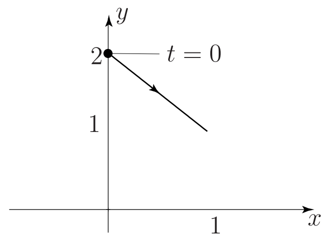
2. Find the tangent line to the parametric curve $x = t^2 - t \; y - t^2 + t$ at the point where $t = 1$.

3. For each of the following curves expressed in parametric form obtain expressions for $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. Use this information to help make a sketch of these curves.

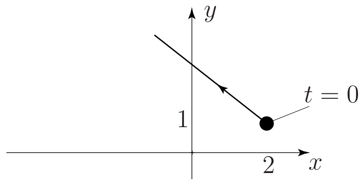
   (a) $x = t^2 - 2t, \quad y = t^2 - 4t$

   (b) $x = t^3 - 3t - 2, \quad y = t^2 - t - 2$

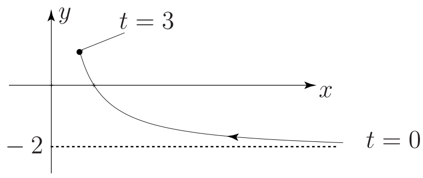
1. (a) $y = 2 - x$



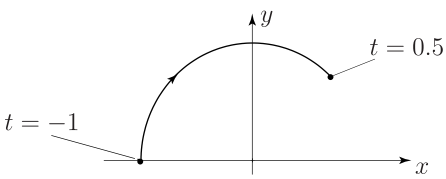
(b) $y = 3 - x$



(c) $y = \dfrac{2}{x} - 2 \qquad \therefore \qquad x(y+2) = 2$



(d) $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

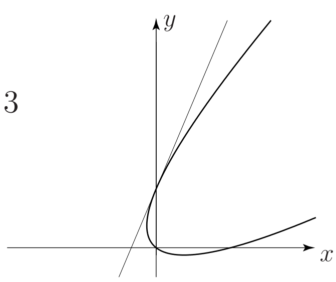


2. $\dfrac{dy}{dt} = 2t + 1 \qquad \dfrac{dx}{dt} = 2t - 1$

$\therefore \qquad \dfrac{dy}{dx} = \dfrac{2t+1}{2t-1} \qquad$ when $t = 1$ then $\dfrac{dy}{dx} = 3$

when $t = 1 \quad x = 0, \quad y = 2$

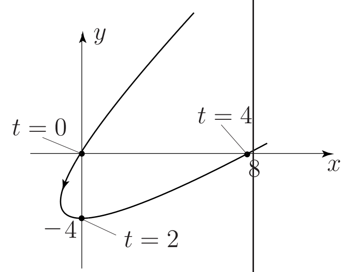
$\therefore \qquad$ tangent line is $y = 3x + 2$



3. (a) $\dfrac{dy}{dt} = 2t - 4 \quad \dfrac{dx}{dt} = 2t - 2$

$$\frac{d^2y}{dt^2} = 2 \qquad \frac{d^2x}{dt} = 2$$

$$\frac{dy}{dx} = \frac{2t-4}{2t-2} = \frac{t-2}{t-1} \qquad \frac{d^2y}{dx^2} = \frac{[(2t-2)-(2t-4)]2}{8(t-1)^3} = \frac{1}{2(t-1)^3}$$



(b) $x = (t-2)(t^2+2t+1) = (t-2)(t+1)^2$

$y = (t+1)(t-2)$

$$\frac{dy}{dt} = 2t - 1 \qquad \frac{dx}{dt} = 3t^2 - 3$$

$$\frac{d^2y}{dt^2} = 2 \qquad \frac{d^2x}{dt^2} = 6t$$

$$\frac{d^2y}{dx^2} = \frac{[2(3t^2-3)-(2t-1)6t]}{(3t^2-3)^3} = \frac{-6t^2+6t-6}{27(t^2-1)^3}$$