Partial Derivatives

18.2

Introduction

When a function of more than one independent input variable changes in one or more of the input variables it is important to calculate the change in the function itself. If we hold all but one of the variables constant and find the rate of change of the function with respect to the remaining variable, then this process is called partial differentiation.
In this section we show how to carry out the process.

Prerequisites

Before starting this Section you should ...

① understand the principle of differentiating a function of one variable

Learning Outcomes

After completing this Section you should be able to ...

- ✓ understand the concept of partial differentiation
- ✓ differentiate a function partially with respect to each of its variables in turn
- ✓ evaluate first partial derivatives
- ✓ carry out two successive partial differentiations
- ✓ formulate second partial derivatives

1. Partial differentiation

The x partial derivative

For a function of a single variable, $y = f(x)$, changing the independent variable x leads to a corresponding change in the dependent variable y. The **rate of change** of y with respect to x is given by the derivative, written $\frac{\mathrm{d}f}{\mathrm{d}x}$. A similar situation occurs with functions of more than one variable. For clarity we shall concentrate on functions of just two variables.
In the relation $z = f(x, y)$ the independent variables are x and y and z is the dependent variable. We have seen in section 1 that as x and y vary the z-value traces out a surface. Now both of the variables x and y may change simultaneously inducing a change in z. However, rather than consider this general situation, to begin with we shall, to begin with, hold one of the independent variables *fixed*. This is equivalent to moving along a curve obtained by intersecting the surface by one of the coordinate planes.

Consider $f(x, y) = x^3 + 2x^2y + y^2 + 2x + 1$. Suppose we keep y constant and vary x; then what is the rate of change of the function f?
Suppose we hold y at the value 3 then

$$f(x, 3) = x^3 + 6x^2 + 9 + 2x + 1.$$

In effect, we now have a function of x only. If we differentiate it with respect to x we obtain the expression:

$$3x^2 + 12x + 0 + 2 + 0 \equiv 3x^2 + 12x + 2.$$

We say that f has been *differentiated partially* with respect to x. We denote the partial derivative of f with respect to x by $\frac{\partial f}{\partial x}$ (to be read as 'partial dee f by dee x'). In this particular example, when $y = 3$:

$$\frac{\partial f}{\partial x} = 3x^2 + 12x + 2.$$

In the same way if y is held at the value 4 then $f(x, 4) = x^3 + 8x^2 + 16 + 2x + 1$ and so, for this value of y

$$\frac{\partial f}{\partial x} = 3x^2 + 16x + 2.$$

If $y = c$, a general constant then

$$f(x, c) = x^3 + 2x^2c + c^2 + 2x + 1$$

and partial differentiation yields the expression

$$\frac{\partial f}{\partial x} = 3x^2 + 4cx + 2.$$

Now if we return to the original formulation

$$f(x, y) = x^3 + 2x^2y + y^2 + 2x + 1$$

and treat y as a constant then the process of partial differentiation with respect to x gives

$$\frac{\partial f}{\partial x} = 3x^2 + 4xy + 0 + 2 = 3x^2 + 4xy + 2.$$

Key Point

The Partial Derivative of f with respect to x

For a function of two variables $z = f(x, y)$ the partial derivative of f with respect to x is denoted by:

$$\frac{\partial f}{\partial x}$$

and is obtained by differentiating $f(x, y)$ with respect to x in the usual way but treating the y-variable (temporarily) as if it were a constant.
Alternative notations are $f_x(x, y)$ and $\frac{\partial z}{\partial x}$.

Example Find $\frac{\partial f}{\partial x}$ for

(a) $f(x, y) = x^3 + x + y^2 + y$, (b) $f(x, y) = x^2y + xy^3$.

Solution

(a) $\frac{\partial f}{\partial x} = 3x^2 + 1 + 0 + 0$ (b) $\frac{\partial f}{\partial x} = 2x.y + 1.y^3$

The y partial derivative

For functions of two variables $f(x, y)$ the x and y variables are on the same footing, so what we have done for the x-variable we can do for the y-variable. We can thus imagine keeping the x-variable fixed and determining the rate of change of f as y changes. This rate of change is denoted by $\frac{\partial f}{\partial y}$.

Key Point

The Partial Derivative of f with respect to y

For a function of two variables $z = f(x, y)$ the partial derivative of f with respect to y is denoted by:

$$\frac{\partial f}{\partial y}$$

and is obtained by differentiating $f(x, y)$ with respect to y in the usual way but treating the x-variable (temporarily) as if it were a constant.
Alternative notations are $f_y(x, y)$ and $\frac{\partial z}{\partial y}$.

Returning to $f(x,y) = x^3 + 2x^2y + y^2 + 2x + 1$ we therefore obtain:

$$\frac{\partial f}{\partial y} = 0 + 2x^2 \times 1 + 2y + 0 + 0 = 2x^2 + 2y.$$

Example Find $\frac{\partial f}{\partial y}$ for

(a) $f(x,y) = x^3 + x + y^2 + y$

(b) $f(x,y) = x^2y + xy^3$

Solution

(a) $\frac{\partial f}{\partial y} = 0 + 0 + 2y + 1$ (b) $\frac{\partial f}{\partial y} = x^2 \times 1 + x \times 3y^2 = x^2 + 3xy^2$

Strictly speaking, we should talk about the partial derivative of f with respect to x and the value of $\frac{\partial f}{\partial x}$ at a specific point e.g. $x = 1,\ y = -2$.

Example Find $f_x(1,-2)$ and $f_y(-3,2)$ for $f(x,y) = x^2 + y^3 + 2xy$

Solution

$f_x(x,y) = 2x + 2y$, so that $f_x(1,-2) = 2 - 4 = -2$

$f_y(x,y) = 3y^2 + 2x$, so that $f_y(-3,2) = 12 - 6 = 6$

Given $f(x,y) = 3x^2 + 2y^2 + xy^3$ find $f_x(1,-2)$ and $f_y(-1,-1)$

First find formulae for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

Your solution

$\frac{\partial f}{\partial x} =$ $\frac{\partial f}{\partial y} =$

$\frac{\partial f}{\partial x} = 6x + y^3$, $\frac{\partial f}{\partial y} = 4y + 3xy^2$

Now find $f_x(1,-2)$ and $f_y(-1,-1)$

Your solution

$f_x(1,-2) =$ $f_y(-1,-1) =$

$f_x(1,-2) = 6 \times 1 + (-2)^3 = -2$

$f_y(-1,-1) = 4 \times (-1) + 3(-1) \times 1 = -7$

Functions of several variables

As we have seen, a function of two variables $f(x, y)$ has two partial derivatives, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. In an exactly analagous way a function of three variables $f(x, y, u)$ will have three partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial u}$ and so on for functions of more than three variables. Each partial derivative is obtained in the same way:

Key Point

The Partial Derivatives of $f(x, y, u, v, w, \ldots)$

For a function of several variables $z = f(x, y, u, v, w, \ldots)$ the partial derivative of f with respect to v (say) is denoted by:

$$\frac{\partial f}{\partial v}$$

and is obtained by differentiating $f(x, y, u, v, w, \ldots)$ with respect to v in the usual way but treating all the other variables (temporarily) as if they were constants.
Alternative notations are $f_v(x, y, u, v, w \ldots)$ and $\frac{\partial z}{\partial v}$.

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial u}$ for $f(x, y, u, v) = x^2 + xy^2 + y^2u^3 - 7uv^4$

Your solution

$\frac{\partial f}{\partial x} =$ $\qquad\qquad$ $\frac{\partial f}{\partial u} =$

$\frac{\partial f}{\partial x} = 2x + y^2 + 0 + 0;$ $\quad \frac{\partial f}{\partial u} = 0 + 0 + y^2 \times 3u^2 - 7v^4 = 3y^2u^2 - 7v^4.$

2. Second partial derivatives

Just as for a function of one variable two successive differentiations (with respect to x) can be carried out and written $\frac{d^2 f}{dx^2}$ so two successive partial differentiations of $f(x, y)$ with respect to x (holding y constant) is denoted by $\frac{\partial^2 f}{\partial x^2}$ (or $f_{xx}(x, y)$) and can be carried out using the same principles as mentioned earlier. That is, we define

$$\frac{\partial^2 f}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

However, for functions of two or more variables other second-order partial derivatives can be obtained. Most obvious is the second derivative of $f(x, y)$ with respect to y is denoted by $\frac{\partial^2 f}{\partial y^2}$ (or $f_{yy}(x, y)$) which is defined as:

$$\frac{\partial^2 f}{\partial y^2} \equiv \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

Example Find $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ for $f(x, y) = x^3 + x^2y^2 + 2y^3 + 2x + y$

Solution

$\frac{\partial f}{\partial x} = 3x^2 + 2xy^2 + 0 + 2 + 0$

$\frac{\partial^2 f}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 6x + 2y^2 + 0 + 0 + 0 = 6x + 2y^2.$

Similarly $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right).$

Now $\frac{\partial f}{\partial y} = 0 + x^2 \times 2y + 6y^2 + 0 + 1 = 2x^2y + 6y^2 + 1$ and $\frac{\partial^2 f}{\partial y^2} = 2x^2 + 12y.$

To evaluate these formulae we can make use of the alternative notation.

Example Find $f_{xx}(-1, 1)$ and $f_{yy}(2, -2)$ for the function of the last example.

Solution

$f_{xx}(-1, 1) = 6 \times (-1) + 2 \times (-1)^2 = -4.$

$f_{yy}(2, -2) = 2 \times (2)^2 + 12 \times (-2) = -16$

Mixed second derivatives

It is possible to carry out a partial differentiation of f with respect to x followed by a partial differentiation with respect to y (or vice-versa). The results are examples of **mixed derivatives**. We must be careful with the notation here, which is as follows:

We use $\frac{\partial^2 f}{\partial x \partial y}$ to mean 'differentiate first with respect to y and then with respect to x' and we use $\frac{\partial^2 f}{\partial y \partial x}$ to mean 'differentiate first with respect to x and then with respect to y'

$$\text{i.e.} \qquad \frac{\partial^2 f}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

Example For $f(x,y) = x^3 + 2x^2y^2 + y^3$ find $\frac{\partial^2 f}{\partial x \partial y}$

Solution

$\frac{\partial f}{\partial y} = 4x^2y + 3y^2$; $\frac{\partial^2 f}{\partial x \partial y} = 8xy$

The final option is to differentiate first with respect to x and then with respect to y i.e. $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$. For the given function

$$\frac{\partial f}{\partial x} = 3x^2 + 4xy^2$$

and

$$\frac{\partial^2 f}{\partial y \partial x} = 8xy.$$

Notice that for this function

$$\frac{\partial^2 f}{\partial x \partial y} \equiv \frac{\partial^2 f}{\partial y \partial x}.$$

This equality of mixed derivatives is true for probably all functions which you are likely to meet in your studies.
To evaluate a mixed derivative we can use the alternative notation. To evaluate $\frac{\partial^2 f}{\partial x \partial y}$ we write $f_{yx}(x,y)$ to indicate that the first differentiation is with respect to y. Similarly, $\frac{\partial^2 f}{\partial y \partial x}$ is denoted by $f_{xy}(x,y)$.

Example Find $f_{yx}(1,2)$ for the function $f(x,y) = x^3 + 2x^2y^2 + y^3$

Solution

$$f_{yx} = 8xy \qquad \text{so} \qquad f_{yx}(1,2) = 8 \times 1 \times 2 = 16.$$

Find $f_{xx}(1,2)$, $f_{yy}(-2,-1)$, $f_{xy}(3,3)$ for

$$f(x,y) \equiv x^3 + 3x^2y^2 + y^2.$$

Your solution

$\frac{\partial f}{\partial x} = 3x^2 + 6xy^2$; $\frac{\partial f}{\partial y} = 6x^2y + 2y$

$\frac{\partial^2 f}{\partial x^2} = 6x + 6y^2$; $\frac{\partial^2 f}{\partial y^2} = 6x^2 + 2$; $\frac{\partial^2 f}{\partial x \partial y} = 12xy = \frac{\partial^2 f}{\partial y \partial x}$

$f_{xx}(1,2) = 6 + 24 = 30$; $f_{yy}(-2,-1) = 26$; $f_{xy}(3,3) = 108$

Exercises

1. For the following functions find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

   (a) $f(x,y) = x + 2y + 3$

   (b) $f(x,y) = x^2 + y^2$

   (c) $f(x,y) = x^3 + xy + y^3$

   (d) $f(x,y) = x^4 + xy^3 + 2x^3y^2$

   (e) $f(x,y,z) = xy + yz$

2. For the functions of Q1 (a) to (d) find $f_x(1,1)$, $f_x(-1,-1)$, $f_y(1,2)$, $f_y(2,1)$.

3. For the functions of Q1 find

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}.$$

4. For the functions of Q1 (a) to (d) find

$$f_{xx}(1,-3), \quad f_{yy}(-2,-2), \quad f_{xy}(-1,1).$$

5. For the following functions find $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x \partial t}$

   (a) $f(x,t) = x\sin(tx) + x^2t$

   (b) $f(x,t,z) = zxt - e^{xt}$

   (c) $f(x,t) = 3\cos(t + x^2)$

Answers

1. (a) $\frac{\partial f}{\partial x} = 1, \quad \frac{\partial f}{\partial y} = 2$

   (b) $\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y$

   (c) $\frac{\partial f}{\partial x} = 3x^2 + y, \quad \frac{\partial f}{\partial y} = x + 3y^2$

   (d) $\frac{\partial f}{\partial x} = 4x^3 + y^3 + 6x^2y^2, \quad \frac{\partial f}{\partial y} = 3xy^2 + 4x^3y$

   (e) $\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x + z$

2.

	$f_x(1,1)$	$f_x(-1,-1)$	$f_y(1,2)$	$f_y(2,1)$
(a)	1	1	2	2
(b)	2	-2	4	2
(c)	4	2	13	5
(d)	11	1	20	38

3. (a) $\frac{\partial^2 f}{\partial x^2} = 0 = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

   (b) $\frac{\partial^2 f}{\partial x^2} = 2 = \frac{\partial^2 f}{\partial y^2}; \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0$

   (c) $\frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial y^2} = 6y; \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 1.$

   (d) $\frac{\partial^2 f}{\partial x^2} = 12x^2 + 12xy^2, \quad \frac{\partial^2 f}{\partial y^2} = 6xy + 4x^3, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 3y^2 + 12x^2y$

   (e) $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 0; \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 1$

4.

	$f_{xx}(1,-3)$	$f_{yy}(-2,-2)$	$f_{xy}(-1,1)$
(a)	0	0	0
(b)	2	2	0
(c)	6	-12	1
(d)	120	-8	15

5. (a) $\frac{\partial f}{\partial x} = \sin(tx) + xt\cos(tx) + 2xt \qquad \frac{\partial^2 f}{\partial t \partial x} = \frac{\partial^2 f}{\partial x \partial t} = 2x\cos(tx) - x^2t\sin(tx) + 2x$

   (b) $\frac{\partial f}{\partial x} = zt - te^{xt} \qquad \frac{\partial^2 f}{\partial t \partial x} = \frac{\partial^2 f}{\partial x \partial t} = z - e^{xt} - txe^{xt}$

   (c) $\frac{\partial f}{\partial x} = -6x\sin(t + x^2) \qquad \frac{\partial^2 f}{\partial t \partial x} = \frac{\partial^2 f}{\partial x \partial t} = -6x\cos(t + x^2)$