

# Stationary Points

# 18.3



## Introduction

The calculation of the optimum value of a function of two variables is a common requirement in many areas of engineering, for example in thermodynamics. Unlike the case of a function of one variable we have to use more complicated criteria to distinguish between the various types of stationary point.



## Prerequisites

Before starting this Section you should ...

- ① have understood the idea of a function of two variables (18.1)
- ② be able to work out partial derivatives (18.2)



## Learning Outcomes

After completing this Section you should be able to ...

- ✓ Identify local maximum points, local minimum points and saddle points on the surface  $z = f(x, y)$
- ✓ Use first partial derivatives to locate the stationary points of a function  $f(x, y)$
- ✓ Use second partial derivatives to determine the nature of a stationary point

# 1. The stationary points of a function of two variables

Figure 1 shows a computer generated picture of the surface defined by the function  $z = x^3 + y^3 - 3x - 3y$ , where both  $x$  and  $y$  take values in the interval  $[-1.8, 1.8]$ .

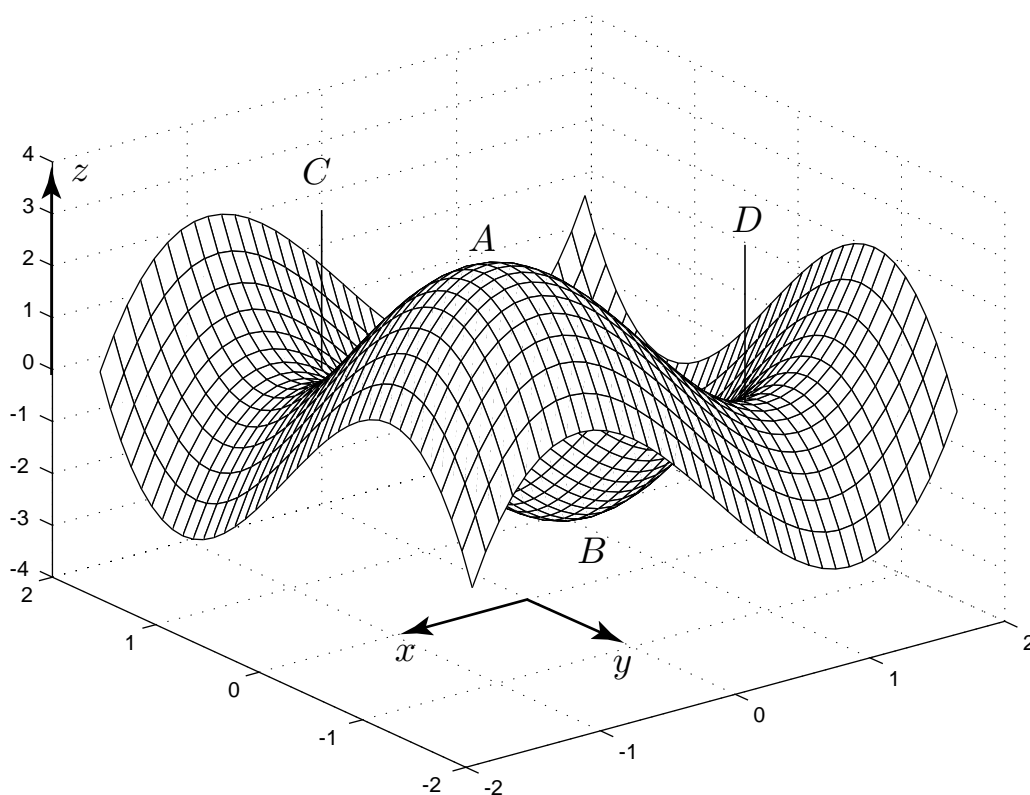


Figure 1

There are four features of particular interest on the surface. At point  $A$  there is a **local maximum**, at  $B$  there is a **local minimum**, and at  $C$  and  $D$  there are what are known as **saddle points**.

At  $A$  the surface is at its greatest height in the immediate neighbourhood, whilst at  $B$  the surface is at its least height in the neighbourhood. If we move on the surface from  $A$  we immediately lose height no matter in which direction we travel. If we move on the surface from  $B$  we immediately gain height, again no matter in which direction we travel.

The features at  $C$  and  $D$  are quite different. In some directions as we move away from these points along the surface we lose height whilst in others we gain height. The similarity in shape to a horse's saddle is evident.

At each point  $P$  of a *smooth* surface one can draw a plane which just touches the surface. This plane is called the **tangent plane** at  $P$ . (The tangent plane is a natural generalisation of the tangent line which can be drawn at each point of a smooth curve). In Figure 1 at each of the points  $A, B, C, D$  the tangent plane to the surface is horizontal at the point of interest. Such points are thus known as **stationary points** of the function. In the next sections we show how to locate stationary points and how to determine their nature using partial differentiation of the function  $f(x, y)$ ,



In Figures 2 and 3 what are the features at  $A$  and  $B$ ?

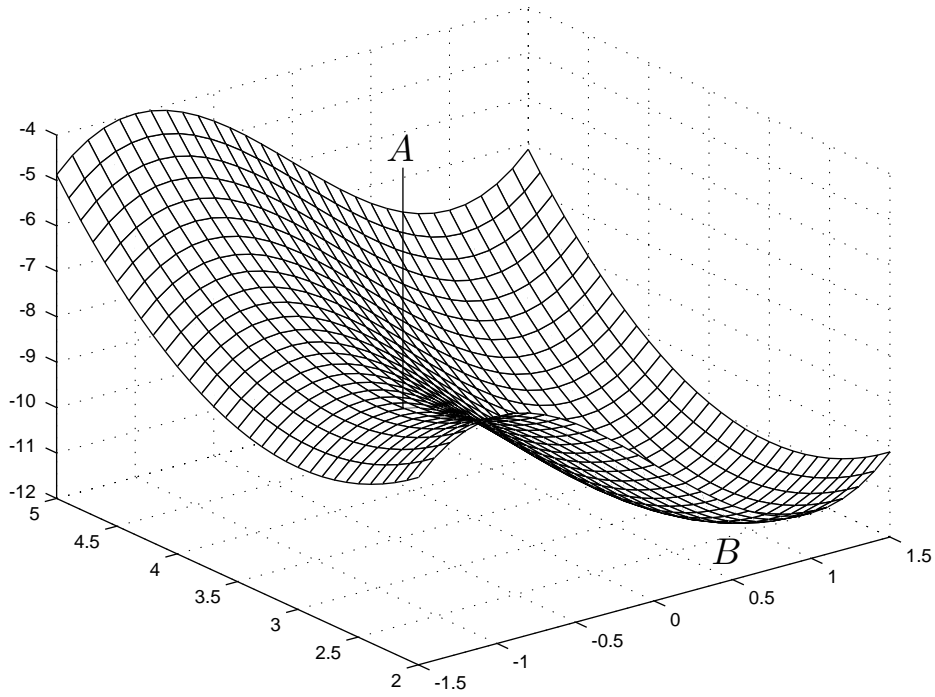


Figure 2

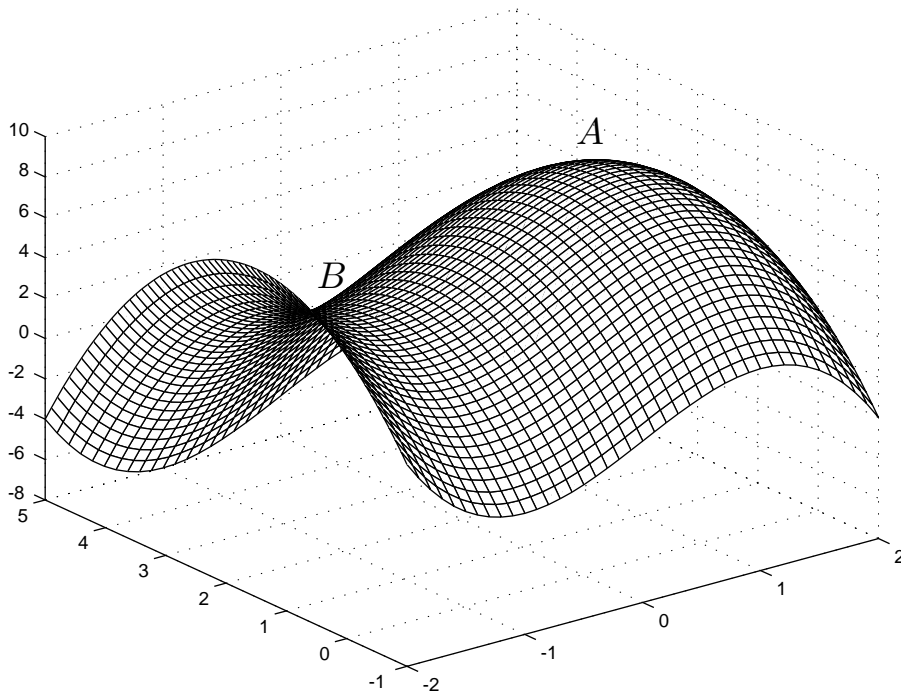


Figure 3

Your solution

Figure 2 A is a saddle point B is a local minimum.  
 Figure 3 A is a local maximum B is a saddle point

## 2. Location of stationary points

As we said in the previous section the tangent plane to the surface  $z = f(x, y)$  is horizontal at a stationary point.

A condition which guarantees that the function  $f(x, y)$  will have a stationary point at a point  $(x_0, y_0)$  is that, at that point both  $\frac{\partial f}{\partial x} = 0$  and  $\frac{\partial f}{\partial y} = 0$ , simultaneously.



Verify that  $(0, 2)$  is a stationary point of the function

$$f(x, y) = 8x^2 + 6y^2 - 2y^3 + 5$$

First, find  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$ .

**Your solution**

$$\frac{\partial f}{\partial x} = 16x \quad ; \quad \frac{\partial f}{\partial y} = 12y - 6y^2$$

Now find the values of these partial derivatives at  $x = 0$ ,  $y = 2$ .

**Your solution**

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 24 - 24 = 0$$

Hence  $(0, 2)$  is a stationary point.

[Note that the **stationary value** is  $f(0, 2) = 0 + 24 - 16 + 5 = 13$ ]

Note also that  $\frac{\partial f}{\partial y} \equiv 6y(2 - y)$ . From this result we note as  $\frac{\partial f}{\partial x} = 0$  when  $x = 0$  and when  $y = 0$ ,  $\frac{\partial f}{\partial y} = 0$  so that  $x = 0$ ,  $y = 0$  i.e.  $(0, 0)$  is a second stationary point of the function.

It is important when solving the simultaneous equations  $\frac{\partial f}{\partial x} = 0$  and  $\frac{\partial f}{\partial y} = 0$  not to 'miss' any solutions. A useful tip is to factorise the left-hand sides and consider systematically all the options.

**Example** Locate the stationary points of

$$f(x, y) = x^4 + y^4 - 36xy$$

### Solution

First we write down the partial derivatives of  $f(x, y)$

$$\begin{aligned}\frac{\partial f}{\partial x} &= 4x^3 - 36y = 4(x^3 - 9y) \\ \frac{\partial f}{\partial y} &= 4y^3 - 36x = 4(y^3 - 9x)\end{aligned}$$

Now we solve the equations  $\frac{\partial f}{\partial x} = 0$  and  $\frac{\partial f}{\partial y} = 0$ :

$$x^3 - 9y = 0 \quad (\text{i})$$

$$y^3 - 9x = 0 \quad (\text{ii})$$

From (ii) we obtain:

$$x = \frac{y^3}{9} \quad (\text{iii})$$

Now substitute into (i)

$$\frac{y^9}{9^3} - 9y = 0$$

$$\text{i.e. } y^9 - 9^4 y = 0$$

$$\text{i.e. } y(y^8 - 3^8) = 0 : \text{ removing the common factor}$$

$$\text{i.e. } y(y^4 - 3^4)(y^4 + 3^4) = 0 : \text{ using the difference of two squares}$$

We therefore obtain, as the only solutions :

$$y = 0 \text{ or } y^4 - 3^4 = 0, \text{ since } y^4 + 3^4 \text{ is never zero}$$

The last equation implies :

$$(y^2 - 9)(y^2 + 9) = 0, \text{ again using the difference of two squares}$$

$$\therefore y^2 = 9 \text{ and } y = \pm 3.$$

Now, using (iii) when  $y = 0$ ,  $x = 0$ , and when  $y = 3$ ,  $x = 3$ : finally when  $y = -3$ ,  $x = -3$ .

The stationary points are  $(0, 0)$ ,  $(-3, -3)$  and  $(3, 3)$ .

Note if we interchange  $x$  and  $y$  in the expression  $f(x, y)$  it is really left as the same expression. The fact that the two coordinates are equal for each of the three stationary points is a reflection of this property.



Locate the stationary points of

$$f(x, y) = x^3 + y^2 - 3x - 6y - 1.$$

First find the partial derivatives of  $f(x, y)$

**Your solution**

$$9 - 4y = \frac{\partial f}{\partial x}, \quad 3 - 2x = \frac{\partial f}{\partial y}$$

Now solve simultaneously the equations  $\frac{\partial f}{\partial x} = 0$  and  $\frac{\partial f}{\partial y} = 0$ .

**Your solution**

giving stationary points at  $(1, 3)$  and  $(-1, 3)$ .  
 Hence  $x = 1$  and  $y = 3$   
 $3 - 2x = 0$  and  $9 - 4y = 0$

### 3. The nature of a stationary point

We quote, without proof, a relatively simple test to determine the nature of a stationary point, once located. If the surface is very flat near the stationary point then the test will not be sensitive enough to determine the nature of the point. The test is dependent upon the values of the second order derivatives:

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y}$$

and also upon a combination of second order derivatives denoted by  $D$  where

$$D \equiv \frac{\partial^2 f}{\partial x^2} \times \frac{\partial^2 f}{\partial y^2} - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 \equiv \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix}$$

The test is as follows:



#### Key Point

##### Testing the stationary points

1. At each stationary point work out the second order partial derivatives.
2. Calculate the value of  $D$  at each of the stationary points.

Then, test each stationary point in turn:

3. If  $D < 0$  the stationary point is a **saddle point**

If  $D > 0$  and  $\frac{\partial^2 f}{\partial x^2} > 0$  the stationary point is a **local minimum**

If  $D > 0$  and  $\frac{\partial^2 f}{\partial x^2} < 0$  the stationary point is a **local maximum**.

If  $D = 0$  then we need an alternative test.

**Example** We have already found that the function:

$$f(x, y) = x^4 + y^4 - 36xy$$

has stationary points at  $(0, 0)$ ,  $(-3, -3)$ ,  $(3, 3)$ . Use the last keypoint to determine which kind of stationary points they are.

**Solution**

We have  $\frac{\partial f}{\partial x} = 4x^3 - 36y$  and  $\frac{\partial f}{\partial y} = 4y^3 - 36x$ .

The  $\frac{\partial^2 f}{\partial x^2} = 12x^2$ ,  $\frac{\partial^2 f}{\partial y^2} = 12y^2$ ,  $\frac{\partial^2 f}{\partial x \partial y} = -36$ .

A tabular presentation is useful.

		Point		
		$(0, 0)$	$(-3, 3)$	$(3, 3)$
Derivatives	$\frac{\partial^2 f}{\partial x^2}$	0	108	108
	$\frac{\partial^2 f}{\partial y^2}$	0	108	108
	$\frac{\partial^2 f}{\partial x \partial y}$	-36	-36	-36
	$D$	< 0	> 0	> 0

Bearing in mind that  $\frac{\partial^2 f}{\partial x^2} > 0$  for the second and third of these points, we see that  $(0, 0)$  is a saddle point whilst  $(-3, 3)$  and  $(3, 3)$  are both local minima.



Determine the nature of the stationary points of  $f(x, y) = x^3 + y^2 - 3x - 6y - 1$ .

Write down the second partial derivatives

**Your solution**

$$\frac{\partial^2 f}{\partial x^2} = 6, \quad \frac{\partial^2 f}{\partial y^2} = 2, \quad \frac{\partial^2 f}{\partial x \partial y} = 0.$$

The stationary points were found in the last guided exercise to be  $(1, 3)$ ,  $(-1, 3)$ .  
Now complete the table below

**Your solution**

		Point	
		$(1, 3)$	$(-1, 3)$
Derivatives	$\frac{\partial^2 f}{\partial x^2}$		
	$\frac{\partial^2 f}{\partial y^2}$		
	$\frac{\partial^2 f}{\partial x \partial y}$		
	$D$		

		$0 >$	$0 <$	$D$
		$0$	$0$	$\frac{\partial^2 f}{\partial x \partial y}$
$2$	$2$	$\frac{\partial^2 f}{\partial y^2}$		
$-6$	$6$	$\frac{\partial^2 f}{\partial x^2}$		
$(-1, 3)$	$(1, 3)$	Point		

What is the nature of each stationary point:



**Your solution**

(1, 3) is a local minimum; (-1, 3) is a saddle point.

For most functions the procedures described above enable us to distinguish between the various types of stationary point. However, note the following example, in which these procedures fail.

$$f(x, y) = x^4 + y^4 + 2x^2y^2.$$

Since  $f(y, x) = f(x, y)$  we expect the coordinates of any stationary point to be equal.

Now

$$\begin{aligned}\frac{\partial f}{\partial x} &= 4x^3 + 4xy^2, & \frac{\partial f}{\partial y} &= 4y^3 + 4x^2y, \\ \frac{\partial^2 f}{\partial x^2} &= 12x^2 + 4y^2, & \frac{\partial^2 f}{\partial y^2} &= 12y^2 + 4x^2, & \frac{\partial^2 f}{\partial x \partial y} &= 8xy\end{aligned}$$

**Location** The stationary points are located where  $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ , that is, where  $4x^3 + 4xy^2 = 0$  and  $4y^3 + 4x^2y = 0$ . A simple factorisation implies  $4x(x^2 + y^2) = 0$  and  $4y(y^2 + x^2) = 0$ . The only solution which satisfies both equations is  $x = y = 0$  and therefore the stationary point is  $(0,0)$ .

**Nature** Unfortunately, all the second partial derivatives are zero at  $(0,0)$  and therefore  $D = 0$ . The test, as described in the keypoint, has failed to give us the necessary information. However, in this example it is easy to see that the stationary point is in fact a local minimum. This could be done by using a computer generated graph of the surface near the point  $(0,0)$ . Alternatively, we observe  $x^4 + y^4 + 2x^2y^2 \equiv (x^2 + y^2)^2$  and we see that  $f(x, y) \geq 0$ ; the only point where  $f(x, y) = 0$  being the stationary point. This is therefore a local (and global) minimum.

## Exercises

1. Determine the nature of the stationary points of the function

$$f(x, y) = 8x^2 + 6y^2 - 2y^3 + 5$$

2. Locate the stationary points of the following functions, and find their nature.

- (a)  $x^3 + 15x^2 - 20y^2 + 10$
- (b)  $4 - x^2 - xy - y^2$
- (c)  $2x^2 + y^2 + 3xy - 3y - 5x + 8$
- (d)  $(x^2 + y^2)^2 - 2(x^2 - y^2) + 1$
- (e)  $x^4 + y^4 + 2x^2y^2 + 2x^2 + 2y^2 + 1$

**Answers**

- 1. (0, 0) local minimum, (0, 2) saddle point.
- 2. (a) (0, 0) saddle point, (-10, 0) local maximum.
- (b) (0, 0) local maximum
- (c) (-1, 3) saddle point
- (d) (0, 0) saddle point, (1, 0) local minimum, (-1, 0) local minimum.
- (e)  $f(x, y) \equiv (x^2 + y^2 + 1)^2$ , local minimum at (0, 0).