Errors and Percentage Change

18.4

Introduction

When one quantity is related to several others by a functional relationship it is possible to estimate the percentage change in it caused by given percentage changes in the other variables. If the input variables are measured and the measurements are in error, due to limits on the precision of measurement, then we can estimate the effect that these errors have on the output.

Prerequisites

Before starting this Section you should ...

① understand the definition of partial derivatives and be able to find them (section 18.1)

Learning Outcomes

After completing this Section you should be able to ...

✓ calculate small errors in a function of more than one variable

✓ calculate approximate values for absolute, relative and percentage relative error

1. Approximations using partial derivatives

Functions of two variables

We saw in section 16.5 how to expand a function of a single variable $f(x)$ in a Taylor series:

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \dots$$

This can be written in the alternative form (by replacing $x - x_0$ by h hence $x = x_0 + h$):

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \dots$$

This expansion can be generalised to functions of two or more variables. Indeed, for functions of two variables we find:

$$f(x_0 + h, \ y_0 + k) \simeq f(x_0, y_0) + hf_x(x_0, y_0) + kf_y(x_0, y_0)$$

where, assuming h and k to be small, we have ignored higher-order terms involving powers of h and k. We define δf to be the change in $f(x, y)$ resulting from small changes to x_0 and y_0. Thus:

$$\delta f = f(x_0 + h, \ y_0 + k) - f(x_0, y_0)$$

and so $\delta f \simeq hf_x(x_0, y_0) + kf_y(x_0, y_0)$. Using the notation δx and δy instead of h and k for small increments in x and y respectively we may write

$$\delta f \simeq \delta x. f_x(x_0, y_0) + \delta y. f_y(x_0, y_0)$$

Finally, using the more common notation for partial derivatives, we write

$$\delta f \simeq \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y.$$

Informally, the term δf is referred to as the **absolute error** in $f(x, y)$ resulting from errors $\delta x, \delta y$ in the variables x and y respectively. Other *measures* of error are used. For example the **relative error** in a variable f is defined as $\frac{\delta f}{f}$ and the **percentage relative error** is $\frac{\delta f}{f} \times 100$.

Key Point

Measures of Error

We define δf to be the change in f at (x_0, y_0) resulting from small changes h, k to x_0 and y_0 respectively. Thus: $\delta f = f(x_0 + h, \ y_0 + k) - f(x_0, y_0)$.

$$\textbf{Absolute error} \text{ in } f \text{ is } \delta f \qquad\qquad \textbf{Relative error} \text{ in } f \text{ is } \frac{\delta f}{f}$$

$$\textbf{Percentage relative error} \text{ in } f \text{ is } \frac{\delta f}{f} \times 100$$

We should note that to determine the error in any particular example we will need to know not only the actual values of δx and δy but also the values of x and y of interest.

Example Estimate the absolute error for the function $f(x, y) = x^2 + x^3 y$

Solution

$\frac{\partial f}{\partial x} = 2x + 3x^2 y; \ \frac{\partial f}{\partial y} = x^3$.

Then $\delta f \simeq (2x + 3x^2 y)\delta x + x^3 \delta y$

We seek the absolute error for $f(x, y) = x^2 y^2 + x + y$. Obtain the absolute error at the point $(-1, 2)$ if $\delta x = 0.1$ and $\delta y = 0.025$.

First find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

Your solution

$$\frac{\partial f}{\partial x} = \qquad\qquad\qquad \frac{\partial f}{\partial y} =$$

$$\frac{\partial f}{\partial x} = 2xy^2 + 1, \quad \frac{\partial f}{\partial y} = 2x^2 y + 1$$

Now obtain the absolute error

Your solution

$$\delta f \simeq (2xy^2 + 1)\delta x + (2x^2 y + 1)\delta y$$

Finally obtain the value of the absolute error at the point of interest.

Your solution

$\delta f \simeq (2xy^2 + 1)\delta x + (2x^2 y + 1)\delta y = (-7)(0.1) + (5)(0.025) = -0.575$. The actual error is easily calculated so we can compare the two values. We find $\delta f = f(x_0 + \delta x, \ y_0 + \delta y) - f(x_0, y_0) = f(-0.9, 2.025) - f(-1, 2) = -0.5534937$. We see that there is a reasonably close correspondence between the two values.

Functions of three or more variables

If f is a function of three or more variables $x, y, u, v, \ldots$ the error induced in f as a result of making small errors $\delta x, \delta y, \delta u, \delta v \ldots$ in $x, y, u, v, \ldots$ is found by a simple generalisation of the expression given above:

$$\delta f \simeq \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial u}\delta u + \frac{\partial f}{\partial v}\delta v + \ldots$$

Example Suppose that the area of triangle ABC is to be calculated by measuring two sides and the included angle. Call the sides b and c and the angle A.
Then the area S of the triangle is given by

$$S = \frac{1}{2}bc\sin A.$$

Now suppose that the side b is measured as 4.00 m, c as 3.00 m and A as 30°. Suppose also that the measurements of the sides could be in error by as much as $\pm$ 0.005m and of the angle by $\pm$ 0.01^o. Calculate the likely error induced in S as a result of the errors in the sides and angle.

Solution

Here S is a function of three variables b, c, A. We calculate

$$S = \frac{1}{2} \times 4 \times 3 \times \frac{1}{2} = 3m^2.$$

Now $\frac{\partial S}{\partial b} = \frac{1}{2}c\sin A$, $\quad \frac{\partial S}{\partial c} = \frac{1}{2}b\sin A$ and $\frac{\partial S}{\partial A} = \frac{1}{2}bc\cos A$. Then

$$\begin{aligned}
\delta S &\simeq \frac{\partial S}{\partial b}\delta b + \frac{\partial S}{\partial c}\delta c + \frac{\partial S}{\partial A}\delta A \\
&= \frac{1}{2}c\sin A \ \delta b + \frac{1}{2}b\sin A \ \delta c + \frac{1}{2}bc\cos A \ \delta A.
\end{aligned}$$

Here $\delta b = \delta c = 0.005$ and $\delta A = \frac{\pi}{180} \times 0.01$ (remember that A must be measured in radians). Substituting these values we see that the error in the calculated value of S is given by the approximation

$$\delta S \simeq \left(\frac{1}{2} \times 3 \times \frac{1}{2}\right) \times 0.005 + \left(\frac{1}{2} \times 4 \times \frac{1}{2}\right) \times 0.005 + \left(\frac{1}{2} \times 4 \times 3 \times \frac{\sqrt{3}}{2}\right) \frac{\pi}{180} \times 0.01$$

$$\simeq \quad 0.0097 \quad m^2$$

Hence the estimated value of S is in error by about $\pm \ 0.01m^2$

The function $f(x, y) = x^2 + y^2 + xy$ is given. Estimate the absolute error in f at the point $x = 2$, $y = 3$ if errors $\pm$ 0.01 and $\pm$ 0.02 are made in x and y respectively.

First find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Your solution

$$\frac{\partial f}{\partial x} = \qquad\qquad\qquad \frac{\partial f}{\partial y} =$$

$$\frac{\partial f}{\partial x} = 2x + y; \qquad \frac{\partial f}{\partial y} = 2y + x.$$

If x has a measured value of 2 and y of 3 calculate the value of $f(x, y)$.

Your solution

$$f(2, 3) = 2^2 + 3^2 + 2 \times 3 = 19.$$

Now since the error in the measured value of x is ± 0.01 and in y is ± 0.02 we have;
$\delta x = 0.01, \quad \delta y = 0.02.$ Write down an approximation to δf.

Your solution

$$\delta f \simeq (2x + y)\delta x + (2y + x)\delta y$$

Calculate the error in f, namely δf

Your solution

$$\delta f \quad \simeq \quad (2 \times 2 + 3) \times 0.01 + (2 \times 3 + 2) \times 0.02$$
$$= \quad 0.07 + 0.16 = 0.23.$$

Hence we quote $f = 19 \mp 0.23$.

2. Percentage relative error

Other measures of error can be obtained from a knowledge of the expression for the absolute error. Suppose that $f(x, y) = x^2 + y^2 + xy$ then

$$\delta f \quad \simeq \quad \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$$

$$= \quad (2x + y)\delta x + (2y + x)\delta y$$

As mentioned earlier the relative error in f is $\frac{\delta f}{f}$ and the percentage relative error is $\left(\frac{\delta f}{f} \times 100\right)$ %.

Hence

$$\frac{\delta f}{f} \simeq \frac{1}{f}\frac{\partial f}{\partial x}\delta x + \frac{1}{f}\frac{\partial f}{\partial y}\delta y$$

$$= \frac{(2x+y)}{x^2+y^2+xy}\delta x + \frac{(2y+x)}{x^2+y^2+xy}\delta y$$

The actual value of the relative error can be obtained once the errors of the independent variables are known and the values of x and y at the point of interest.

In the special case where the function is a combination of powers of the input variables then we have a short cut to finding the relative error in the value of the function. For example if $f(x,y,u) = \frac{x^2 y^4}{u^3}$ then

$$\frac{\partial f}{\partial x} = \frac{2xy^4}{u^3}, \qquad \frac{\partial f}{\partial y} = \frac{4x^2 y^3}{u^3}, \qquad \frac{\partial f}{\partial u} = -\frac{3x^2 y^4}{u^4}$$

Hence

$$\delta f \simeq \frac{2xy^4}{u^3}\delta x + \frac{4x^2 y^3}{u^3}\delta y - \frac{3x^2 y^4}{u^4}\delta u$$

Finally,

$$\frac{\delta f}{f} \simeq \frac{2xy^4}{u^3} \times \frac{u^3}{x^2 y^4}\delta x + \frac{4x^2 y^3}{u^3} \times \frac{u^3}{x^2 y^4}\delta y - \frac{3x^2 y^4}{u^4} \times \frac{u^3}{x^2 y^4}\delta u$$

Cancelling down the fractions,

$$\frac{\delta f}{f} \simeq 2\frac{\delta x}{x} + 4\frac{\delta y}{y} - 3\frac{\delta u}{u} \qquad (*)$$

so that

rel. error in $f \simeq 2$(rel. error in x)+ 4(rel. error in y) $-$ 3(rel. error in u).

Note that if we write

$$f(x,y,u) = x^2 y^4 u^{-3}$$

we see that the coefficients of the relative errors on the right-hand side of $(*)$ are the powers of the appropriate variable.

To find the percentage relative error we simply multiply the relative error by 100.

If $f = \frac{x^3 y}{u^2}$ and x, y, u are subject to percentage relative errors of 1, -1 and 2 respectively find the approximate percentage relative error in f.

First find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial u}$.

Your solution

$$\frac{\partial f}{\partial x} = \qquad\qquad \frac{\partial f}{\partial y} = \qquad\qquad \frac{\partial f}{\partial u} =$$

$$-\frac{2x^3 y}{u^3} = \frac{\partial f}{\partial u}, \quad \frac{x^3}{u^2} = \frac{\partial f}{\partial y}, \quad \frac{3x^2 y}{u^2} = \frac{\partial f}{\partial x}.$$

Now write down an expression for δf

Your solution

$\delta f \simeq$

$$\delta f \approx \frac{3x^2}{u^2}\delta x + \frac{x^3}{u^2}\delta y - \frac{2x^3 y}{u^3}\delta u$$

Hence write down an expression for the percentage relative error in f

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Your solution

$$\frac{\delta f}{f} \times 100 \quad \approx \quad \frac{3x^2}{u^2}\frac{u^2}{x^3 y}\delta x \times 100 + \frac{x^3}{u^2} \times \frac{u^2}{x^3 y}\delta y \times 100 - \frac{2x^3 y}{u^3} \times \frac{u^2}{x^3 y}\delta u \times 100$$

$$\approx \quad 3\frac{\delta x}{x} \times 100 + \frac{\delta y}{y} \times 100 - 2\frac{\delta u}{u} \times 100$$

$$= \quad (3(1) - 1 - 2(2))$$

$$= \quad -2\%$$

Note that $f = x^3 y u^{-2}$.

Exercises

1. The sides of a right angled triangle enclosing the right angle are measured as 6m and 8m respectively. The maximum errors in each measurement are $\pm$ 0.1m. Find the maximum error in the calculated area.

2. In question 1, the angle opposite the 8m side is calculated from $\tan \theta = \frac{8}{6}$ as $53^0 8'$. Calculate the approximate maximum error in that angle.

3. If $v = \sqrt{\dfrac{3x}{y}}$ find the maximum percentage error in v due to errors of 1% in x and 3% in y.

4. If $n = \dfrac{1}{2L}\sqrt{\dfrac{E}{d}}$ and L, E and d can be measured correct to 1%, how accurate is the calculated value of n?

5. The area of a segment of a circle which subtends an angle θ is given by $A = \frac{1}{2}r^2(\theta - \sin \theta)$. The radius r is measured with a maximum percentage error of 0.2% and θ is measured as 45^0 with a maximum error of 0.1^0. Find the maximum percentage error in the calculated area.

Answers

1. $A = \frac{1}{2}xy$ $\delta A \approx \frac{\partial A}{\partial x}\delta x + \frac{\partial A}{\partial y}\delta y$ $\delta A \approx \frac{y}{2}\delta x + \frac{x}{2}\delta y$

   Maximum error $= |y\delta x| + |x\delta y| = 0.7$ sq m.

2. $\theta = \tan^{-1}\frac{y}{x}$ so $\delta\theta = \frac{\partial\theta}{\partial x}\delta x + \frac{\partial\theta}{\partial y}\delta y = -\frac{y}{x^2+y^2}\delta x + \frac{x}{x^2+y^2}\delta y$

   Maximum error in θ is $\left|\frac{-8}{6^2+8^2}(0.1)\right| + \left|\frac{6}{6^2+8^2}(0.1)\right| = 0.014$ rad. This is $0.8°$.

3. Take logarithms of both sides:

   $\ln v = \frac{1}{2}\ln 3 + \frac{1}{2}\ln x - \frac{1}{2}\ln y$ so $\frac{\delta v}{v} \approx \frac{\delta x}{2x} - \frac{\delta y}{2y}$

   Maximum percentage error in $v = \left|\frac{\delta x}{2x}\right| + \left|-\frac{\delta y}{2y}\right| = \frac{1}{2}\% + \frac{3}{2}\% = 2\%$.

4. Take logarithms of both sides:

   $\ln n = -\ln 2 - \ln L + \frac{1}{2}\ln E - \frac{1}{2}\ln d$ so $\frac{\delta n}{n} = -\frac{\delta L}{L} + \frac{\delta E}{2E} - \frac{\delta d}{2d}$

   Maximum percentage error in $n = \left|-\frac{\delta L}{L}\right| + \left|\frac{\delta E}{2E}\right| + \left|-\frac{\delta d}{2d}\right| = 1\% + \frac{1}{2}\% + \frac{1}{2}\% = 2\%$.

5. $A = \frac{1}{2}r^2(\theta - \sin\theta)$ so $\frac{\delta A}{A} = \frac{2\delta r}{r} + \frac{1-\cos\theta}{\theta - \sin\theta}\delta\theta$

   $\frac{\delta A}{A} = 2(0.2)\% + \left\{\frac{1 - \frac{1}{\sqrt2}}{\frac{\pi}{4} - \frac{1}{\sqrt2}}\right\}\times 100\% \times \frac{\pi}{1800} = (0.4 + 0.65)\% = 1.05\%$