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differential equations

Learning outcomes

In this workbook you will learn what a differential equation is and how to recognise some of the basic different types. You will learn how to apply some common techniques used to obtain general solutions of differential equations and how to fit initial or boundary conditions to obtain a unique solution. You will appreciate how differential equations arise in applications and you will gain some experience in applying your knowledge to model a number of engineering problems using differential equations.

Time allocation

You are expected to spend approximately twelve hours of independent study on the material presented in this workbook. However, depending upon your ability to concentrate and on your previous experience with certain mathematical topics this time may vary considerably.

Modelling with Differential Equations 19.1

Introduction

Many models of engineering systems involve the rate of change of a quantity. There is thus a need to incorporate derivatives into the mathematical model. These mathematical models are examples of differential equations.
Accompanying the differential equation will be one or more conditions that let us obtain a unique solution. Often we solve the differential equation first to obtain a general solution; then we apply the conditions to obtain the unique solution. It is important to know which conditions must be specified in order to obtain a unique solution.

Prerequisites

Before starting this Section you should ...

① be able to differentiate; (Workbook 11)

② be able to integrate; (Workbook 13)

Learning Outcomes

After completing this Section you should be able to ...

✓ understand the use of differential equations in modelling engineering systems

✓ identify the order and type of a differential equation

✓ recognise the nature of a general solution

✓ determine the nature of the appropriate additional conditions which will give a unique solution to the equation

1. Case Study: Newton's Law of Cooling

When a hot liquid is placed in a cooler environment, experimental observation shows that its temperature decreases to approximately that of its surroundings. A typical graph of the temperature of the liquid plotted against time is shown in Figure 1.

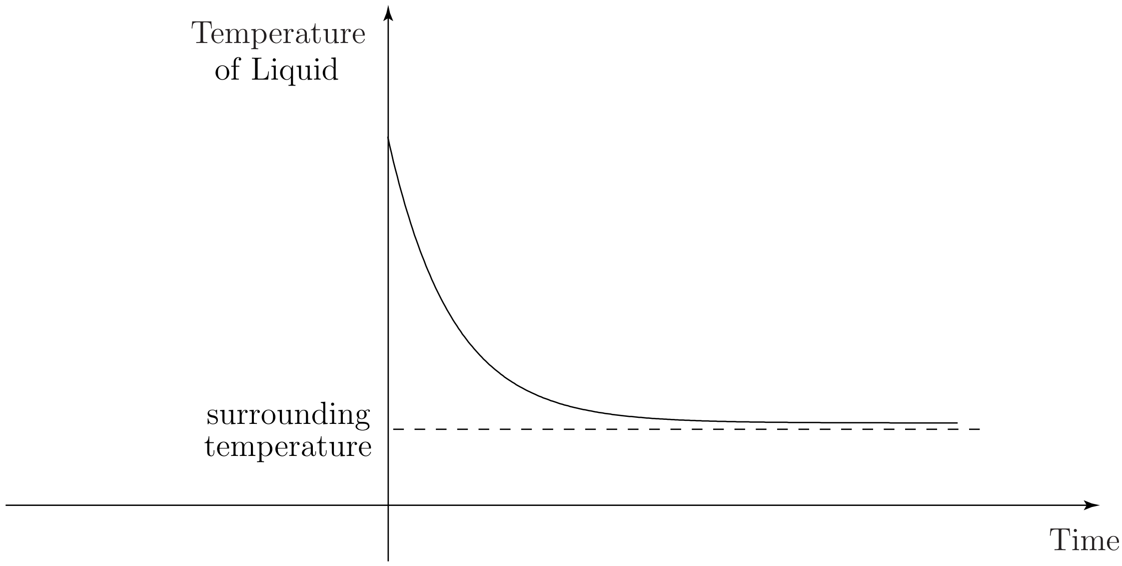


Figure 1.

After an initially rapid decrease the temperature changes progressively less rapidly and eventually the curve appears to 'flatten out'.

Newton's law of cooling states that the rate of cooling of liquid is proportional to the difference between its temperature and the temperature of its environment.

To convert this into mathematics, let t be the time elapsed (in seconds, s), θ the temperature of the liquid (^{o}C),and θ_0 the temperature of the liquid at the start ($t = 0$). The temperature of the surroundings is denoted by $\theta_{\rm s}$.

Write down the mathematical equation which is equivalent to Newton's law and state the accompanying condition.

Your solution

First, find expressions for the rate of cooling and the difference between the liquid's temperature and that of the environment.

The rate of cooling is the rate of change of temperature with time: $\frac{d\theta}{dt}$. The temperature difference is $\theta - \theta_{\rm s}$.

Your solution

Now formulate Newton's law of cooling.

You should obtain $\frac{\mathrm{d}\theta}{\mathrm{d}t} \propto (\theta - \theta_{\mathrm{s}})$ or, equivalently: $\frac{\mathrm{d}\theta}{\mathrm{d}t} = -k(\theta - \theta_{\mathrm{s}})$. k is a constant of proportion and the negative sign is present because $(\theta - \theta_{\mathrm{s}})$ is positive, whereas $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ must be negative, since θ decreases with time. The units of k are s^{-1}. The accompanying condition is $\theta = \theta_0$ at $t = 0$ which simply states the temperature of the liquid when the cooling begins.

In the above example we call t the independent variable and θ the dependent variable. Since the condition is given at $t = 0$ we refer to it as an initial condition. For reference, the solution of the differential equation which satisfies the initial condition is $\theta = \theta_{\mathrm{s}} + (\theta_0 - \theta_{\mathrm{s}})\mathrm{e}^{-kt}$.

2. The General Solution of a Differential Equation

Consider the formula $y = A\mathrm{e}^{2x}$ where A is an arbitrary constant. If we differentiate it we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2A\mathrm{e}^{2x}$$

and so, since $y = A\mathrm{e}^{2x}$ we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2y.$$

Then the differential equation satisfied by y is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2y.$$

Notice that we have eliminated the arbitrary constant.
Now consider the formula

$$y = A\cos 3x + B\sin 3x$$

where A and B are arbitrary constants. Differentiating, we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -3A\sin 3x + 3B\cos 3x.$$

Differentiating a second time gives

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -9\,A\cos 3x - 9\,B\sin 3x.$$

The right-hand side is simply (-9) times the expression for y. Hence y satisfies the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -9y.$$

Find a differential equation satisfied by $y = A\cosh 2x + B\sinh 2x$ where A and B are arbitrary constants.

Your solution

Differentiating once we obtain

$$\frac{dy}{dx} = 2A\sinh 2x + 2B\cosh 2x$$

Differentiating a second time we obtain

$$\frac{d^2y}{dx^2} = 4A\cosh 2x + 4B\sinh 2x$$

Hence

$$\frac{d^2y}{dx^2} = 4y$$

We have seen that an expression including one arbitrary constant required one differentiation to obtain a differential equation which eliminated the arbitrary constant. Where two constants were present, two differentiations were required. Is the converse true? For example, would a differential equation involving $\frac{dy}{dx}$ as the only derivative have a general solution with one arbitrary constant and would a differential equation which had $\frac{d^2y}{dx^2}$ as the highest derivative produce a general solution with two arbitrary constants? The answer is, usually, yes.

Integrate twice the differential equation

$$\frac{d^2y}{dx^2} = \frac{w}{2}(\ell x - x^2),$$

where w and ℓ are constants, to find a general solution for y.

Your solution

Integrating once:

$$\frac{dy}{dx} = \frac{w}{2}\left(\frac{\ell x^2}{2} - \frac{x^3}{3}\right) + A$$

where A is an arbitrary constant (of integration). Integrating a second time:

$$y = \frac{w}{2}\left(\frac{\ell x^3}{6} - \frac{x^4}{12}\right) + Ax + B$$

where B is a second arbitrary constant.

Consider the simple differential equation

$$\frac{dy}{dx} = 2x.$$

On integrating, we obtain the general solution

$$y = x^2 + C \qquad (1)$$

where C is an arbitrary constant. As C varies we get different solutions, each of which belongs to the family of solutions (1). Figure 2 shows some examples.

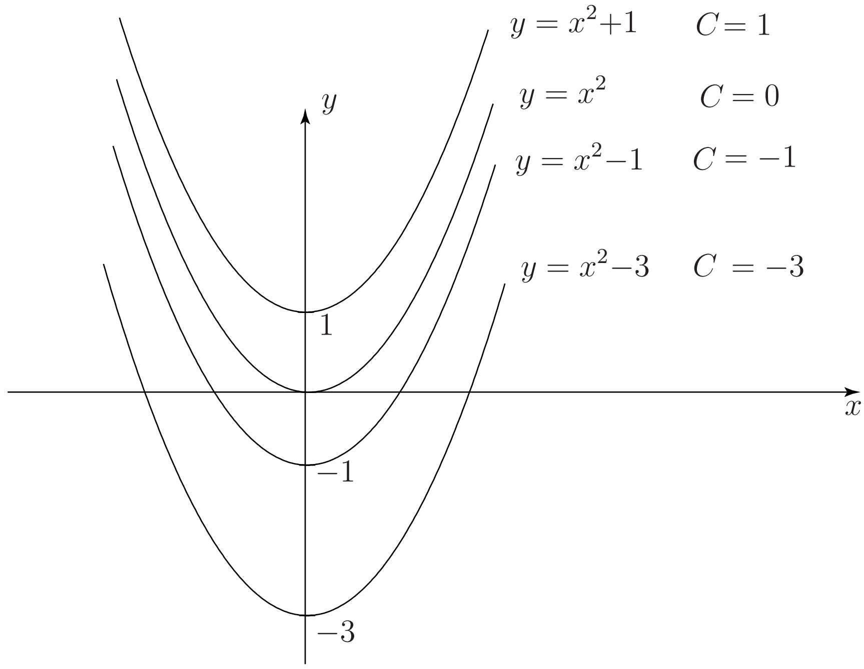


Figure 2.

It can be shown that no two members of this family of graphs ever meet and that through each point in the x-y plane passes one, and only one, of these graphs.
Hence if we specify the boundary condition $y = 2$ when $x = 0$, written $y(0) = 2$, then using (1):

$$2 = 0 + C \quad \text{so that} \quad C = 2$$

and $y = x^2 + 2$ is the unique solution.

Find the unique solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$ which satisfies the condition $y(1) = 4$.

Your solution

You should obtain $y = x^3 + 3$ since, by a single integration we have $y = x^3 + C$, where C is an arbitrary constant. Now when $x = 1$, $y = 4$ so that $4 = 1 + C$. Hence $C = 3$ and the unique solution is $y = x^3 + 3$.

Example Solve $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x$.

Solution

Integrating the differential equation once produces $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + A$.. The general solution is found by integrating a second time to give $y = x^3 + Ax + B$, where A and B are arbitrary constants. If we impose the conditions $y(0) = 2$ and $y(1) = 3$ then, at $x = 0$, we have $y = 2 = 0 + 0 + B = B$ so that $B = 2$. Also, at $x = 1$, we have $y = 3 = 1 + A + B = 1 + A + 2$. Therefore $A = 0$ and the solution is

$$y = x^3 + 2.$$

Had the second condition been $y(1) = 5$ (instead of $y(1) = 3$) then at $x = 1$

$$y = 5 = 1 + A + 2 \qquad \text{so that} \quad A = 2$$

and so the solution in this case is

$$y = x^3 + 2x + 2.$$

Finally if the second condition had been

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \text{ at } x = 0 \qquad \text{i.e.} \qquad y'(0) = 1$$

then since $y'(x) = 3x^2 + A$, and putting $x = 0$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 = 0 + A$$

so that $A = 1$ and the solution in this case is $y = x^3 + x + 2$.

3. Classifying a Differential Equation

When solving differential equations (either analytically or numerically) it is important to be able to recognise the various kinds that can arise. We therefore need to introduce some terminology which will help us to distinguish one kind of differential equation from another. The **order** of a differential equation is the order of the highest derivative in the equation. A differential equation is said to be **linear** if the dependent variable and its derivatives occur to the first power only and if there are no products involving the dependent variable and/or its derivatives.

Example Classify the differential equations specifying order and type (linear/non-linear)

(a) $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = x^2$ (b) $\dfrac{d^2x}{dt^2} = \left(\dfrac{dx}{dt}\right)^3 + 3x$ (c) $\dfrac{dx}{dt} - x = t^2$

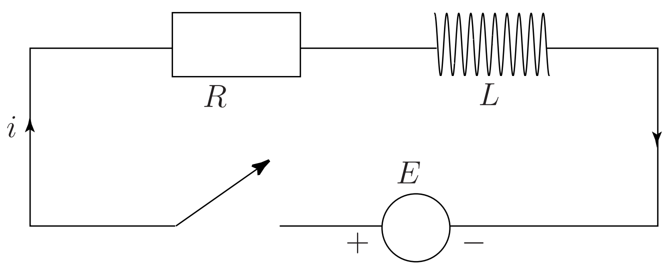
(d) $\dfrac{dy}{dt} + \cos y = 0$ (e) $\dfrac{dy}{dt} + y^2 = 4$

Solution

(a) Second order, linear. (b) Second order. Because of the cubic term the equation is non-linear. (c) First order, linear. (d) First order. The equation is non-linear because of the $\cos y$ term. (e) As (d). This time it is the y^2 term which causes the equation to be non-linear. Note that in (a) the independent variable is x whereas in the other cases it is t. In (a), (d) and (e) the dependent variable is y and in (b) and (c) it is x.

Exercises

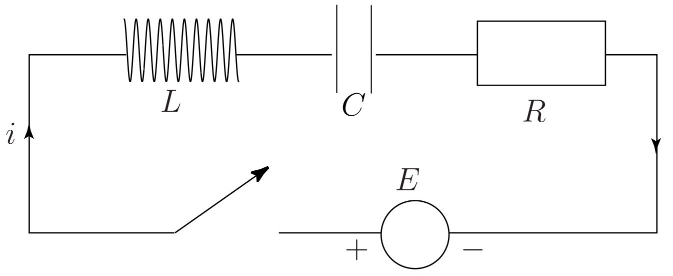
1. The figure shows an RL circuit.



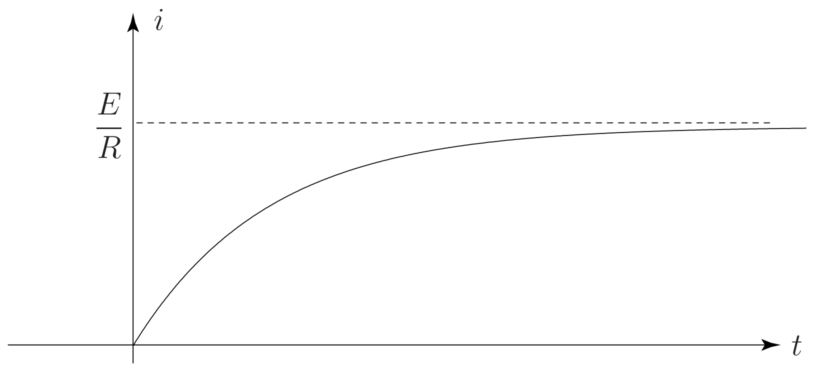
The switch is closed at $t = 0$ and a constant voltage E is then applied to the circuit. The voltage across the resistor is iR where i is the current flowing in the circuit and R is the (constant) resistance. The voltage across the inductance is $L\frac{di}{dt}$ where L is the constant inductance.
Kirchhoff's law of voltages states that the applied voltage is the sum of the other voltages in the circuit. Write down a differential equation for the current i and state the initial condition.

2. In the LCR circuit below the voltage across the capacitor is q/C where q is the charge on the capacitor, and C is the capacitance. Note that $\frac{dq}{dt} = i$. Find a differential equation for i and

write down the initial conditions if the initial charge is zero and the switch is closed at $t = 0$.



3. The figure below shows the graph of i against t (from exercise 1).



What information does this graph convey?
4. Find a differential equation satisfied by
(a) $y = A\cos 4x + B\sin 4x$
(b) $x = A\,\mathrm{e}^{-2t}$
(c) $y = A\sin x + B\sinh x + C\cos x + D\cosh x$ (harder)
5. Find the family of solutions of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2x.$$

Sketch the curves of some members of the family on the same axes.
What is the solution if $y(1) = 3$?
6. Find the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2.$$

Find the solutions which satisfy
(a) $y(0) = 2,\ y(1) = 8$
(b) $y(0) = 1,\ y'(0) = -2.$
7. Classify the differential equations
(a) $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} + 3\,\dfrac{\mathrm{d}x}{\mathrm{d}t} = x$ (b) $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} = \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \dfrac{\mathrm{d}y}{\mathrm{d}x}$
(c) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = \sin x$ (d) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2.$

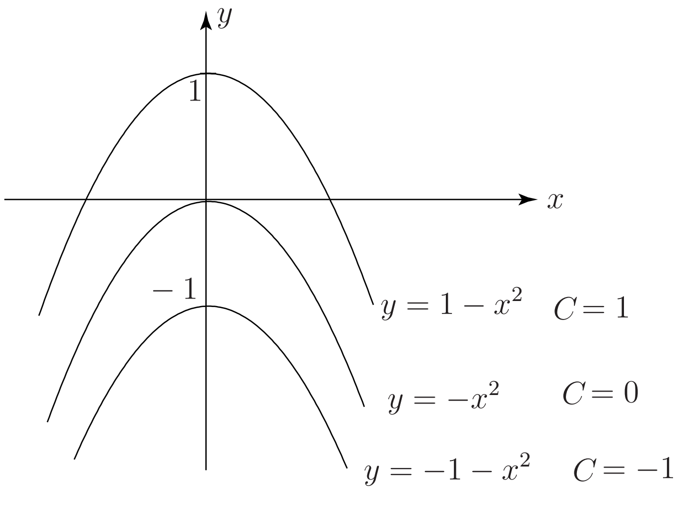
Answers 1. $L\,\frac{di}{dt} + Ri = E$; $i = 0$ at $t = 0$.

2. $L\,\frac{d^2q}{dt^2} + R\,\frac{dq}{dt} + \frac{q}{C} = E$; $q = 0$ and $i = \frac{dq}{dt} = 0$ at $t = 0$.

3. Current increases rapidly at first, then less rapidly and tends to the value $\frac{E}{R}$ which is what it would be in the absence of L.

4. (a) $\frac{d^2y}{dx^2} = -16y$ (b) $\frac{dx}{dt} = -2x$ (c) $\frac{d^4y}{dx^4} = y$

5. $y = -x^2 + C$



If $3 = -1 + C$ then $C = 4$ and $y = -x^2 + 4$.

6. $y = x^4 + Ax + B$ (a) When $x = 0$, $y = 2 = B$; hence $B = 2$. When $x = 1$, $y = 8 = 1 + A + B = 3 + A$ hence $A = 5$ and $y = x^4 + 5x + 2$. (b) When $x = 0$ $y = 1 = B$. Hence $B = 1$; $\frac{dy}{dx} = y' = 4x^3 + A$, so at $x = 0$, $y' = -2 = A$. Therefore $y = x^4 - 2x + 1$

7. (a) Second order, linear (b) Third order, non-linear *(squared term)* (c) First order, linear (d) Second order, non-linear *(Product term)*