Applications of Differential Equations 19.4

Introduction

Sections 19.2 and 19.3 have introduced several techniques for solving commonly-occurring first-order and second-order ordinary differential equations. In this Section we solve a number of these equations which model engineering systems.

Prerequisites

Before starting this Section you should ...

① understand what is meant by a differential equation; (section 19.1)

② be familiar with the terminology associated with differential equations: order, dependent variable and independent variable; (section 19.1)

③ be able to integrate; (sections 14.1-14.8)

④ have completed sections 19.2, 19.4, 19.5 and 19.6

Learning Outcomes

After completing this Section you should be able to ...

✓ recognise and solve first-order ordinary differential equations, modelling simple electrical circuits, projectile motion and Newton's law of cooling

✓ recognise and solve second-order ordinary differential equations with constant coefficients modelling free electrical and mechanical oscillations

✓ recognise and solve second-order ordinary differential equations with constant coefficients modelling forced electrical and mechanical oscillations

1. Modelling with First-order Equations

Applying Newton's law of cooling

In section 19.1 we introduced Newton's law of cooling. The model equation was

$$\frac{d\theta}{dt} = -k(\theta - \theta_s) \qquad (1)$$

where $\theta = \theta(t)$ is the temperature of the cooling object at time t, θ_s the temperature of the environment (assumed constant) and k is a thermal constant related to the object. Let θ_0 be the initial temperature of the liquid, i.e.

$$\theta = \theta_0 \quad \text{at} \quad t = 0.$$

Solve this initial value problem.

Your solution

Separate the variables to obtain an equation connecting two integrals

$$\int \frac{d\theta}{\theta - \theta_s} = -\int k\, dt$$

Your solution

Now integrate both sides of this equation

$\ln(\theta - \theta_s) = -kt + C$ where C is constant

Your solution

Apply the initial condition and take exponentials to obtain a formula for θ

$\ln(\theta_0 - \theta_s) = C.$

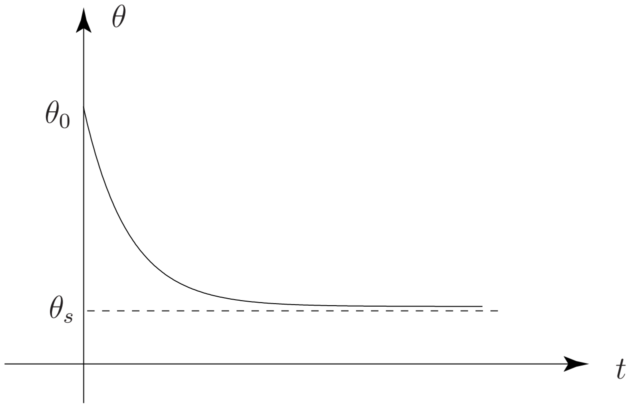
Hence $\quad \ln(\theta - \theta_s) = -kt + \ln(\theta_0 - \theta_s) \quad$ so that $\quad \ln(\theta - \theta_s) - \ln(\theta_0 - \theta_s) = -kt$

Thus, rearranging and inverting, we find:

$$\ln\left(\frac{\theta - \theta_s}{\theta_0 - \theta_s}\right) = -kt \qquad \therefore \qquad \frac{\theta - \theta_s}{\theta_0 - \theta_s} = e^{-kt}$$

and so, finally, $\theta = \theta_s + (\theta_0 - \theta_s)e^{-kt}$.

The graph of θ against t is shown in the figure below.



We see that as time increases ($t \to \infty$), then the temperature of the object cools down to that of the environment, that is: $\theta \to \theta_s$.
Note that we could have solved (1) by the integrating factor method.

Write the equation as

$$\frac{d\theta}{dt} + k\,\theta = k\theta_s \qquad (2)$$

What is the integrating factor for this equation?

Your solution

$e^{\int k\,dt} = e^{kt}$ is the integrating factor.

Multiplying (2) by this factor we find that

$$e^{kt}\frac{d\theta}{dt} + ke^{kt}\theta = k\theta_s e^{kt} \qquad \text{or, rearranging,} \qquad \frac{d}{dt}(e^{kt}\theta) = k\theta_s e^{kt}$$

Your solution

Now integrate this equation and apply the initial condition

Integration produces $e^{kt}\theta = \theta_s e^{kt} + C$, where C is an arbitrary constant. Then, applying the initial condition: when $t = 0$, $\theta_0 = \theta_s + C$ so that $C = \theta_0 - \theta_s$ and, finally,

$$\theta = \theta_s + (\theta_0 - \theta_s)e^{-kt},$$

as before.

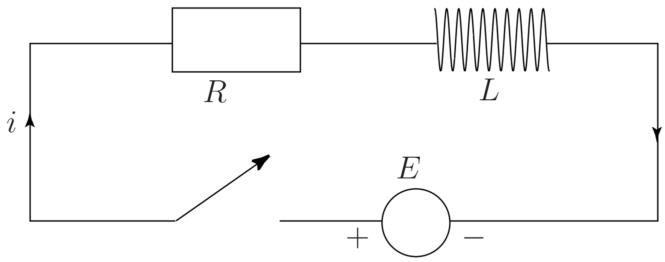
Electrical circuits

Another application of first-order differential equations arises in the modelling of electrical circuits.

In Section 19.1 the differential equation for the RL circuit of the figure below was shown to be

$$L\,\frac{\mathrm{d}i}{\mathrm{d}t} + Ri = E$$

in which the initial condition is $i = 0$ at $t = 0$.



Write this equation in standard form $\{\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)\}$ and obtain the integrating factor.

Your solution

Divide the differential equation through by L to obtain

$$\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{R}{L}\,i = \frac{E}{L}$$

which is now in standard form. The integrating factor is $\mathrm{e}^{\int \frac{R}{L}\mathrm{d}t} = \mathrm{e}^{Rt/L}$.

Multiplying the equation in standard form by the integrating factor gives

$$\mathrm{e}^{Rt/L}\frac{\mathrm{d}i}{\mathrm{d}t} + \mathrm{e}^{Rt/L}\frac{R}{L}i = \frac{E}{L}\mathrm{e}^{Rt/L}$$

or, rearranging,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{Rt/L}\ i) = \frac{E}{L}\mathrm{e}^{Rt/L}.$$

Your solution

Now integrate both sides and apply the initial condition to obtain the solution

You should obtain $i = \frac{E}{R}(1 - e^{-Rt/L})$, since integrating the differential equation gives:

$$e^{Rt/L}\, i = \frac{E}{R} e^{Rt/L} + C$$

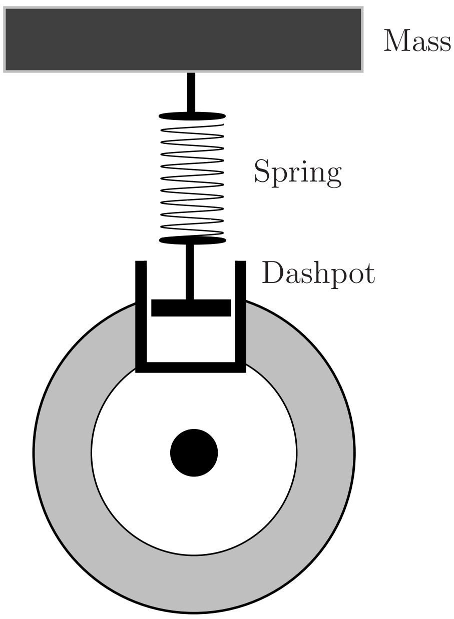
where C is a constant. Then applying the initial condition $i = 0$ when $t = 0$ gives

$$0 = \frac{E}{R} + C$$

so that $C = -\frac{E}{R}$ and $e^{Rt/L}\, i = \frac{E}{R} e^{Rt/L} - \frac{E}{R}$. Finally, $i = \frac{E}{R}(1 - e^{-Rt/L})$. Note that as $t \to \infty$, $i \to \frac{E}{R}$ so as t increases the effect of the inductor diminishes to zero.

2. Modelling Free Mechanical Oscillations

Consider the following schematic diagram of a shock absorber:



The equation of motion can be described in terms of the vertical displacement x of the mass. Let m be the mass, $k\dfrac{\mathrm{d}x}{\mathrm{d}t}$ the damping force resulting from the dashpot and nx the restoring force resulting from the spring. Here, k and n are constants. Then the equation of motion is

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -k\frac{\mathrm{d}x}{\mathrm{d}t} - nx.$$

Suppose that the mass is displaced a distance x_0 initially and released from rest. Then at $t = 0$, $x = x_0$ and $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 0$. Writing the differential equation in standard form:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + k\frac{\mathrm{d}x}{\mathrm{d}t} + nx = 0.$$

We shall see that the nature of the oscillations described by this differential equation, depends crucially upon the relative values of the mechanical constants m, k and n.

Find the auxiliary equation of this differential equation and solve it.

Your solution

Put $x = e^{\lambda t}$ then the auxiliary equation is

$$m\lambda^2 + k\lambda + n = 0.$$

Hence $\lambda = \dfrac{-k \pm \sqrt{k^2 - 4mn}}{2m}$.

The value of k controls the amount of damping in the system. We explore the solution for two particular values of k.

No damping

If $k = 0$ then there is no damping. We expect, in this case, that once motion has started it will continue for ever. The motion that ensues is called simple harmonic motion. In this case we have

$$\lambda = \frac{\pm\sqrt{-4mn}}{2m},$$

that is,

$$\lambda = \pm\sqrt{\frac{n}{m}}\,\mathrm{i} \qquad \text{where} \quad \mathrm{i} = \sqrt{-1}.$$

In this case the solution for the displacement x is:

$$x = A\cos\left(\sqrt{\frac{n}{m}}\ t\right) + B\sin\left(\sqrt{\frac{n}{m}}\ t\right)$$

where A, B are arbitrary constants.

Your solution

Now apply the initial conditions to find the unique solution:

You should obtain $x = x_0\cos\left(\sqrt{\frac{n}{m}}\ t\right)$. To see this we first need an expression for the derivative, $\frac{dx}{dt}$:

$$\frac{dx}{dt} = -\sqrt{\frac{n}{m}}\,A\sin\left(\sqrt{\frac{n}{m}}\ t\right) + \sqrt{\frac{n}{m}}\,B\cos\left(\sqrt{\frac{n}{m}}\ t\right)$$

When $t = 0$, $\frac{dx}{dt} = 0$ so that

$$\sqrt{\frac{n}{m}}\ B = 0 \qquad \text{so that} \qquad B = 0.$$

Therefore

$$x = A\cos\left(\sqrt{\frac{n}{m}}\ t\right).$$

Imposing the remaining initial condition: when $t = 0$, $x = x_0$ so that $x_0 = A$ and finally:

$$x = x_0\cos\left(\sqrt{\frac{n}{m}}\ t\right).$$

Light damping

If $k^2 - 4mn < 0$, i.e. $k^2 < 4mn$ then the roots of the auxiliary equation are complex:

$$\lambda_1 = \frac{-k + i\sqrt{4mn - k^2}}{2m} \qquad \lambda_2 = \frac{-k - i\sqrt{4mn - k^2}}{2m}$$

Then, after some rearrangement:

$$x = \mathrm{e}^{-kt/2m}\left[A\cos pt + B\sin pt\right]$$

in which $p = \sqrt{4mn - k^2}/2m$.

If $m = 1,\ n = 1$ and $k = 1$ find λ_1 and λ_2 and then find the solution for the displacement x.

Your solution

$\lambda_1 =$ $\lambda_2 =$

$\therefore \quad x =$

$\lambda = \frac{-1+i\sqrt{4-1}}{2} = -1/2 \pm i\sqrt{3}/2$. Hence $x = e^{-t/2}\left[A\cos\frac{\sqrt{3}}{2}t + B\sin\frac{\sqrt{3}}{2}t\right]$.

Impose the initial conditions $x = x_0,\ \frac{dx}{dt} = 0$ at $t = 0$ to find the arbitrary constants.

Your solution

Differentiating, we obtain

$$\frac{dx}{dt} = -\frac{1}{2}e^{-t/2}\left[A\cos\frac{\sqrt{3}}{2}t + B\sin\frac{\sqrt{3}}{2}t\right] + e^{-t/2}\left[-\frac{\sqrt{3}}{2}A\sin\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{2}B\cos\frac{\sqrt{3}}{2}t\right]$$

At $t = 0$,

$$x = x_0 = A \qquad \text{(i)}$$

$$\frac{dx}{dt} = 0 = -\frac{1}{2}A + \frac{\sqrt{3}}{2}B \qquad \text{(ii)}$$

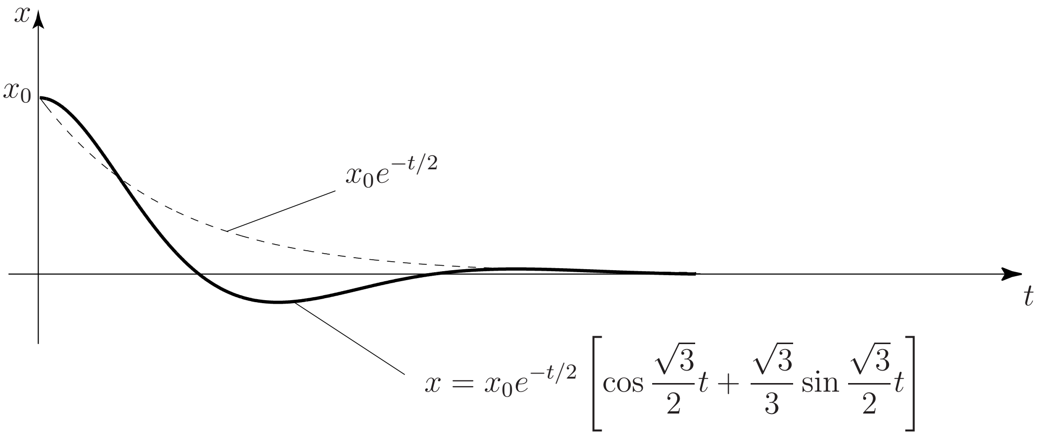
Solving (i) and (ii) we obtain

$$A = x_0 \qquad B = \frac{\sqrt{3}}{3}x_0$$

Then

$$x = x_0e^{-t/2}\left[\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{3}\sin\frac{\sqrt{3}}{2}t\right].$$

The graph of x against t is shown below. This is the case of light damping. As the damping in the system decreases (i.e. $k \to 0$) the number of oscillations (in a given time interval) will increase. In many mechanical systems these oscillations are usually unwanted and the designer would choose a value of k to either reduce them or to eliminate them altogether. For the choice $k^2 = 4mn$, known as the critical damping case, all the oscillations are absent.



Heavy damping

If $k^2 - 4mn > 0$, i.e. $k^2 > 4mn$ then there are two real roots of the auxiliary equation, λ_1 and λ_2:

$$\lambda_1 = \frac{-k + \sqrt{k^2 - 4mn}}{2m} \qquad \lambda_2 = \frac{-k - \sqrt{k^2 - 4mn}}{2m}$$

Then

$$x = A\mathrm{e}^{\lambda_1 t} + B\mathrm{e}^{\lambda_2 t}.$$

If $m = 1,\ n = 1$ and $k = 2.5$ find λ_1 and λ_2 and then find the solution for the displacement x.

Your solution

$\lambda_1 =$ $\qquad\qquad$ $\lambda_2 =$

$\therefore \quad x =$

$$\lambda = \frac{-2.5 \pm \sqrt{6.25 - 4}}{2} = -1.25 \pm 0.75$$

Hence $\lambda_1,\ \lambda_2 = -0.5, -2$ and so $x = Ae^{-0.5t} + Be^{-2t}$

Impose the initial conditions $x = x_0$, $\frac{dx}{dt} = 0$ at $t = 0$ to find the arbitrary constants.

Your solution

Differentiating, we obtain

$$\frac{dx}{dt} = -0.5Ae^{-0.5t} - 2Be^{-2t}$$

At $t = 0$,

$$x = x_0 = A + B \qquad \text{(i)}$$

$$\frac{dx}{dt} = 0 = -0.5A - 2B \qquad \text{(ii)}$$

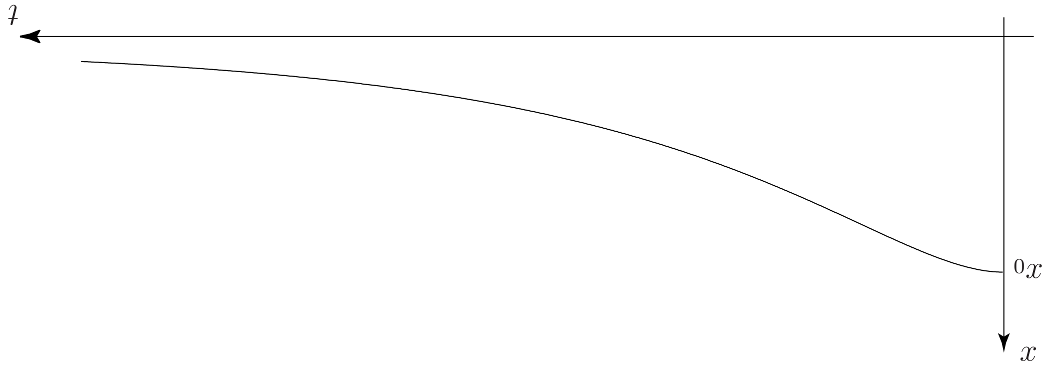
Solving (i) and (ii) we obtain

$$A = \frac{4}{3}x_0 \qquad B = -\frac{1}{3}x_0$$

Then

$$x = \frac{1}{3}x_0(4e^{-0.5t} - e^{-2t}).$$

The graph of x against t is shown below. This is the case of heavy damping. Other cases are dealt with in the Exercises at the end of the section.



3. Modelling Forced Mechanical Oscillations

Suppose now that the mass is subject to a force $f(t)$ after the initial disturbance. Then the equation of motion is

$$m\frac{d^2x}{dt^2} + k\,\frac{dx}{dt} + nx = f(t)$$

Consider the case $f(t) = F\cos\omega t$, that is, an oscillatory force of magnitude F and angular frequency ω. Choosing specific values for the constants in the model: $m = n = 1$, $k = 0$, and $\omega = 2$ we find

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x = F\cos 2t$$

Find the complementary function for this equation.

Your solution

The homogeneous equation is

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x = 0$$

with auxiliary equation $\lambda^2 + 1 = 0$. Hence the complementary function is

$$x_{\mathrm{cf}} = A\cos t + B\sin t.$$

Now find a particular integral for the differential equation.

Your solution

Say $x_{\mathrm{p}} = C\cos 2t + D\sin 2t$ so that $\frac{\mathrm{d}^2x_{\mathrm{p}}}{\mathrm{d}t^2} = -4C\cos 2t - 4D\sin 2t$.. Substituting into the differential equationgives

$$(-4C + C)\cos 2t + (-4D + D\sin 2t) \equiv F\cos 2t.$$

Comparing coefficients gives $-3C = F$ and $-3D = 0$ so that $D = 0$, $C = -\frac{1}{3}F$ and $x_{\mathrm{p}} = -\frac{1}{3}F\cos 2t$. The general solution of the differential equation is therefore

$$x = x_{\mathrm{p}} + x_{\mathrm{cf}} = -\frac{1}{3}F\cos 2t + A\cos t + B\sin t.$$

Finally, apply the initial conditions to find the solution for the displacement x.

Your solution

You should obtain

$$x = -\frac{1}{3}F\cos 2t + \left(x_0 + \frac{1}{3}F\right)\cos t.$$

To obtain this we need to determine the derivative and apply the initial conditions:

$$\frac{dx}{dt} = \frac{2}{3}F\sin 2t - A\sin t + B\cos t.$$

At $t = 0$

$$x = x_0 = -\frac{1}{3}F + A \qquad \text{(i)}$$

$$\frac{dx}{dt} = 0 = B \qquad \text{(ii)}$$

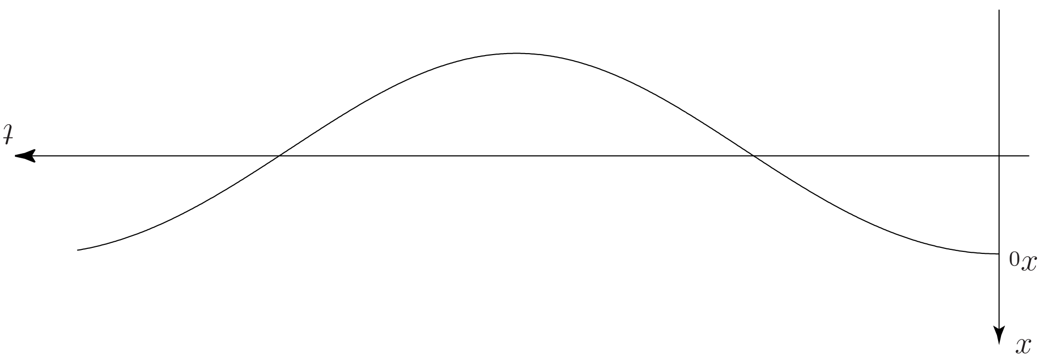
Hence

$$B = 0 \qquad \text{and} \qquad A = x_0 + \frac{1}{3}F.$$

Then

$$x = -\frac{1}{3}F\cos 2t + \left(x_0 + \frac{1}{3}F\right)\cos t.$$

The graph of x against t is shown below.



If the angular frequency ω of the applied force is nearly equal to that of the free oscillation the phenomenon of **beats** occurs. If the angular frequencies are equal we get the phenomenon of **resonance**. Note that we can eliminate resonance by introducing damping into the system.

Exercises

1. In an RC circuit (a resistor and a capacitor in series) the applied emf is a constant E. Given that $\frac{\mathrm{d}q}{\mathrm{d}t} = i$ where q is the charge in the capacitor, i the current in the circuit, R the resistance and C the capacitance the equation for the circuit is

$$Ri + \frac{q}{C} = E.$$

If the initial charge is zero find the charge subsequently.

2. If the voltage in the RC circuit is $E = E_0 \cos \omega t$ find the charge and the current at time t.

3. An object is projected from the Earth's surface. What is the least velocity (the escape velocity) of projection in order to escape the gravitational field, ignoring air resistance.
The equation of motion is

$$m\,v\frac{\mathrm{d}v}{\mathrm{d}x} = -m\,g\frac{R^2}{x^2}$$

where the mass of the object is m, its distance from the centre of the Earth is x and the radius of the Earth is R.

4. The radial stress p at distance r from the axis of a thick cylinder subjected to internal pressure is given by $p + r\frac{\mathrm{d}p}{dr} = A - p$ where A is a constant. If $p = p_0$ at the inner wall $r = r_1$ and is negligible $(p = 0)$ at the outer wall $r = r_2$ find an expression for p.

5. The equation for an LCR circuit with applied voltage E is

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri + \frac{1}{C}q = E.$$

By differentiating this equation find the solution for $q(t)$ and $i(t)$ if $L = 1$, $R = 100$, $C = 10^{-4}$ and $E = 1000$ given that $q = 0$ and $i = 0$ at $t = 0$.

6. Consider the free vibration problem when $m = 1,\ n = 1$ and $k = 2$. (Critical damping)
Find the solution for $x(t)$.

7. Repeat problem 6 for the case
$m = 1,\ n = 1$ and $k = 1.5$ (light damping)

8. Consider the forced vibration problem with $m = 1,\ n = 25,\ k = 8,\ E = \sin 3t,\ x_0 = 0$ with an initial velocity of 3.

Answers 1. Use the equation in the form $R\frac{\mathrm{d}q}{\mathrm{d}t}+\frac{q}{C}=E$ or $\frac{\mathrm{d}q}{\mathrm{d}t}+\frac{1}{RC}q=\frac{E}{R}$. The integrating factor is $\mathrm{e}^{t/RC}$ and the general solution is

$$q = EC(1-\mathrm{e}^{-t/RC}) \qquad \text{and as} \qquad t\to\infty \quad q \to EC.$$

2. $q = \dfrac{E_0C}{1+\omega^2R^2C^2}\left[\cos\omega t - \mathrm{e}^{-t/RC} + \omega RC\sin\omega t\right]$

$i = \dfrac{\mathrm{d}q}{\mathrm{d}t} = \dfrac{E_0C}{1+\omega^2R^2C^2}\left[-\omega\sin\omega t + \dfrac{1}{RC}\,\mathrm{e}^{-t/RC} + \omega^2RC\cos\omega t\right].$

3. $v_{\min} = \sqrt{2gR}$. If $R = 6378$ km and $g = 9.81$ m s^{-2} then $v_{\min} = 11.2$ km s^{-1}.

4. $p = \dfrac{p_0r_1^2}{r_1^2-r_2^2}\left(1-\dfrac{r_2^2}{r^2}\right)$

5. $q = 0.1 - \dfrac{1}{10\sqrt{3}}\mathrm{e}^{-50t}(\sin 50\sqrt{3}t + \sqrt{3}\cos 50\sqrt{3}t) \qquad i = \dfrac{20}{\sqrt{3}}\mathrm{e}^{-50t}\sin 50\sqrt{3}t.$

6. $x = x_0(1+t)e^{-t}$

7. $x = x_0e^{-0.75t}(\cos\frac{\sqrt{7}}{4}t + \frac{3}{\sqrt{7}}\sin\frac{\sqrt{7}}{4}t)$

8. $x = \dfrac{1}{104}\left[\mathrm{e}^{-4t}\left(3\cos 3t + 106\sin 3t\right) - 3\cos 3t + 2\sin 3t\right]$