The graph of a function and parametric form

2.2

Introduction

Engineers often find mathematical ideas easier to understand when these are portrayed visually as opposed to algebraically. Graphs are a convenient and widely-used way of portraying functions. By inspecting a graph it is easy to describe a number of properties of a function. For example, where is the function positive, and where is it negative? Where is it increasing and where is it decreasing? Do function values repeat? Questions like these can be answered once the graph of a function has been drawn. In this section we will describe how the graph of a function is obtained and introduce various terminology associated with graphs.

We have seen in Section 2.1 that it is possible to represent a function using the form $y = f(x)$. An alternative representation is to write expressions for both y and x in terms of a third variable known as a **parameter**. The variables t or θ are normally used to denote the parameter.

For example, when a projectile such as a ball or rocket is thrown or launched, the x and y coordinates of its path can be described by a function in the form $y = f(x)$. However, it is often useful to also give its x coordinate as a function of the time after launch, that is $x(t)$, and the y coordinate similarly as $y(t)$. Here time t is the parameter.

Prerequisites

Before starting this Section you should ...

① understand what is meant by a function

Learning Outcomes

After completing this Section you should be able to ...

✓ draw the graph of a variety of functions

✓ explain what is meant by the domain and range of a function

1. The graph of a function

Consider the function $f(x) = 2x$. The output is obtained by multiplying the input by 2. We can choose several values for the input to this function and calculate the corresponding outputs. We have done this for integer values of x between -2 and 2 and the results are shown in Table 1.

Table 1

input, x	-2	-1	0	1	2
output, $f(x)$	-4	-2	0	2	4

To construct the graph of this function we first draw a pair of **axes** - a vertical axis and a horizontal axis. These are drawn at right-angles to each other and intersect at the **origin** as shown in Figure 1.

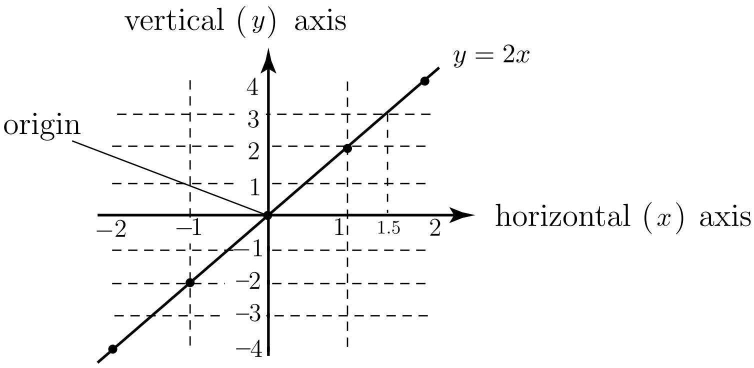


Figure 1. The two axes intersect at the origin.

Each pair of input and output values can be represented on a graph by a single point. The input values are measured along the horizontal axis and the output values are measured along the vertical axis. The horizontal axis is often called the x axis. The vertical axis is commonly referred to as the y axis so that we often write the function as

$$y = f(x) = 2x$$

or simply

$$y = 2x$$

Each pair of x and y values in the table is plotted as a single point, shown as $\bullet$ in Figure 1. A general point is often labelled as (x, y). The values x and y are said to be the **coordinates** of the point. The points are then joined with a smooth curve to produce the required graph as shown in Figure 1. Note that in this case the graph is a straight line. The graph can then be used to find function values other than those given in the table. For example, directly from the graph we can see that when $x = 1.5$, the value of y is 3.

Draw up a table of values of the function $f(x) = x^3$ for x between -3 and 3. Use the Table to plot a graph of this function.

Your solution

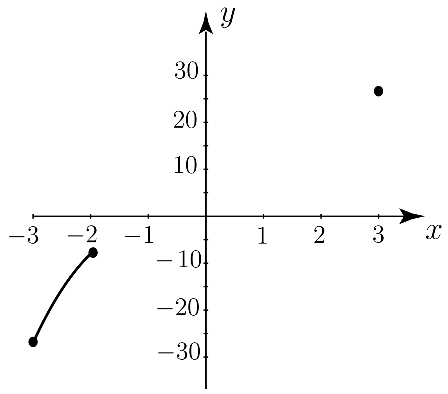
Complete the following table.

input, x	-3	-2	-1	0	1	2	3
output, $f(x)$	-27	-8					27

input, x	-3	-2	-1	0	1	2	3
output, $f(x)$	-27	-8	-1	0	1	8	27

Your solution

Now add your points to the graph of $f(x) = x^3$ and draw a smooth curve through them.



Dependent and independent variables

Since x and y can have a number of different values they are variables. Here x is called the **independent variable** and y is called the **dependent variable**. Knowing or choosing a value of the independent variable x, the function rule enables us to calculate the corresponding value of the dependent variable y. To show this dependence we often write $y(x)$. This is read as 'y is a function of x' or 'y depends upon x', or simply 'y of x'. Note that it is the independent variable which is the input to the function and the dependent variable which is the output.

The domain and range of a function

The set of values which we allow the independent variable to take is called the **domain** of the function. A domain is often an interval on the x axis. For example, the function

$$y = g(x) = 5x + 2, \qquad -5 \le x \le 20$$

has any value of x between -5 and 20 inclusive as its domain because it has been stated as this. If the domain of a function is not stated then it is taken to be the largest set possible. For example

$$h(t) = t^2 + 1$$

has domain $-\infty < x < \infty$ since h is defined for every value of t and the domain has not been stated otherwise.

Later, you will meet some functions for which certain values of the independent variable must be excluded from the domain because at these values the function would be undefined. One such example is $f(x) = \frac{1}{x}$ for which we must exclude the value $x = 0$, since $\frac{1}{0}$ is a meaningless quantity. Similarly, we must exclude the value $x = 2$ from the domain of $f(x) = \frac{1}{x-2}$.
The set of values of the function for a given domain, that is, the set of y values, is called the **range** of the function. The range of $g(x)$ (above) is $-23 \leq g(x) \leq 102$ and the range of $h(t)$ is $1 \leq h(t) < \infty$, although this may not be apparent to you at this stage. Usually the range of a function can be identified quite easily by inspecting its graph.

Example Consider the function given by $g(t) = 2t^2 + 1$, $-2 \leq t \leq 2$.
a) State the domain of the function.
b) Plot a graph of the function.
c) Deduce the range of the function from the graph.

Solution

a) The domain is given as the interval $-2 \leq t \leq 2$, that is any value of t between -2 and 2 inclusive.

b) To construct the graph a table of input and output values must be constructed first . See Table 2.

Table 2

t	-2	-1	0	1	2
$y = g(t)$	9	3	1	3	9

Each pair of t and y values in the table is plotted as a single point shown as • in Figure 2. The points are then joined with a smooth curve to produce the required graph.

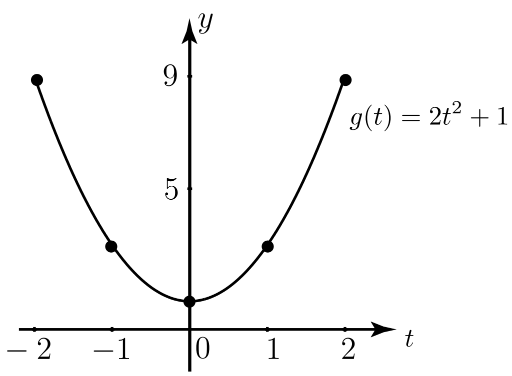


Figure 2. Graph of $g(t) = 2t^2 + 1$

c) The range is the set of values which the function takes. By inspecting the graph we see that the range of g is the interval $1 \leq g(t) \leq 9$.

Consider the function given by $f(x) = x^2 + 2, \quad -3 \le x \le 3$

a) State the domain of the function.

b) Draw up a table of input and output values for this function.

c) Plot a graph of the function.

d) Deduce the range of the function by inspecting the graph.

Your solution

a) Recall that the domain of a function $f(x)$ is the set of values that x is allowed to take. Write down this set of values:

$-3 \le x \le 3$

Your solution

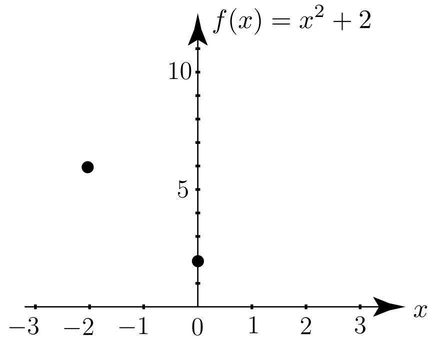
b) The table of values has been partially calculated. Complete this now:

input, x	-3	-2	-1	0	1	2	3
output, $x^2 + 2$		6		2			

x	-3	-2	-1	0	1	2	3
$x^2 + 2$	11	6	3	2	3	6	11

Your solution

c) Part of the graph $f(x) = x^2 + 2$ is shown in the figure. Complete it.



Your solution

d) Recall that the range of the function is the set of values that the function takes as x is varied. It is possible to deduce this from the graph. Write this set as an interval.

d) $[2, 11]$

Exercises

1. Explain the meaning of the terms 'dependent variable' and 'independent variable'. When plotting a graph, which variables are plotted on which axes ?
2. When stating the coordinates of a point, which coordinate is given first ?
3. Explain the meaning of an expression such as $y(x)$ in the context of functions. What is the interpretation of $x(t)$?
4. Explain the meaning of the terms 'domain' and 'range' when applied to functions.
5. Plot a graph of the following functions. In each case state the domain and the range of the function.
a) $f(x) = 3x + 2, \quad -2 \le x \le 5$
b) $g(x) = x^2 + 4, \quad -2 \le x \le 3$
c) $p(t) = 2t^2 + 8, \quad -2 \le t \le 4$
d) $f(t) = 6 - t^2, \quad 1 \le t \le 5$
6. Explain why the value $x = -7$ should be excluded from the domain of $f(x) = \frac{5}{x+7}$.
7. What value(s) should be excluded from the domain of $f(t) = \frac{1}{t^2}$?

Answers 1. The independent variable is plotted on the horizontal axis.
2. The independent variable is given first, as in (x, y).
3. $x(t)$ means that the dependent variable x is a function of the independent variable t.
5. a) domain $[-2, 5]$, range $[-4, 17]$. b) $[-2, 3]$, $[4, 13]$, c) $[-2, 4]$, $[8, 40]$, d) $[1, 5]$, $[-19, 5]$. 6. f is undefined when $x = -7$. 7. $t = 0$.

2. Parametric representation of a function

Suppose we write x and y in terms of t in the form

$$x = 4t \qquad y = 2t^2, \qquad \text{for } -1 \le t \le 1 \qquad (1)$$

For different values of t between -1 and 1, we can calculate pairs of values of x and y. For example when $t = 1$ we see that $x = 4(1) = 4$ and $y = 2 \times 1^2 = 2$. That is $t = 1$ corresponds to the point with (x, y) coordinates $(4, 2)$.
A table of values is given in Table 1.

t	-1	-0.5	0	.5	1
x	-4	-2	0	2	4
y	2	0.5	0	0.5	2

If the resulting points are plotted on a graph then different values of t correspond to different points on the graph. The graph of (1) is plotted in Figure 1.

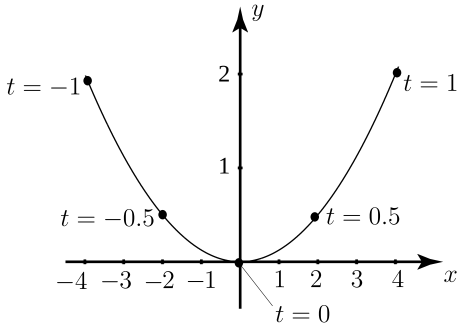


Figure 3. Graph of the function defined parametrically by $x = 4t$, $y = 2t^2$, $-1 \leq t \leq 1$.

It is often possible to convert a parametric representation of a function into the more usual form by combining the two expressions to eliminate the parameter. Thus if $x = 4t$ we can write $t = \frac{x}{4}$ and so

$$\begin{aligned} y = 2t^2 &= 2\left(\frac{x}{4}\right)^2 \\ &= \frac{2x^2}{16} \\ &= \frac{x^2}{8} \end{aligned}$$

Using $y = \frac{x^2}{8}$ we can, by giving x values, find corresponding values of y. Plotting these (x, y) values gives, of course, exactly the same curve as in Figure 1.

Consider the function $x = \frac{1}{2}\left(t + \frac{1}{t}\right)$, $y = \frac{1}{2}\left(t - \frac{1}{t}\right)$, $1 \leq t \leq 8$.
a) Draw up a table of values of this function.
b) Plot a graph of the function

Your solution

a) A partially completed table of values has been prepared. Complete the table.

t	1	2	3	4	5	6	7	8
x	1	1.25	1.67					4.06
y	0	0.75						3.94

t	1	2	3	4	5	6	7	8
x	1	1.25	1.67	2.13	2.60	3.08	3.57	4.06
y	0	0.75	1.33	1.88	2.40	2.92	3.43	3.94

Your solution

b) The graph is shown in Figure 2. Add your points to those already marked on the graph.

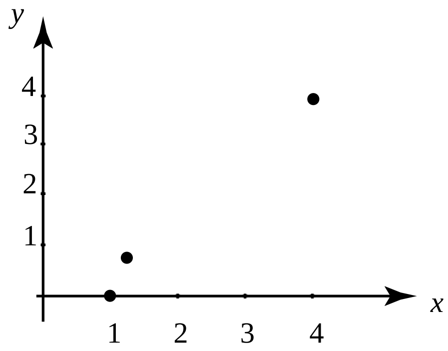


Figure 2

It is possible to eliminate t between the two equations so that the original parametric form can be expressed as $x^2 - y^2 = 1$.

A particle with mass m falls under gravity so that at time t its distance from the y-axis is $2t$ and its distance from the x-axis is $-mg\frac{t^2}{2} + 3$ where g is a constant (the acceleration due to gravity). Find the value of t when the particle crosses the x-axis and, at this time, find the distance from the y-axis.

Begin by obtaining the parametric equations of the path of the particle.

Your solution

$x =$ $\qquad$ $y =$

$x = 2t$ $\qquad$ $y = -mg\dfrac{t^2}{2} + 3$

Now find the value of t when $y = 0$

Your solution

$t =$

$t = \sqrt{6/(mg)}$

Finally, obtain the value of x at this value of t

Your solution

$$x =$$

$x = 2\sqrt{6/(mg)}$

Exercises

1. Explain what is meant by the term ‘parameter’.
2. Consider the parametric equations $x = +\sqrt{t}$, $y = t$, for $t \geq 0$.

a) Draw up a table of values of t, x and y for values of t between 0 and 10.
b) Plot a graph of this function.
c) Obtain an explicit equation for y in terms of x.

Answers 2. c) $y = x^2$, $0 \leq x \leq \sqrt{10}$