Characterising functions

2.4

Introduction

There are a number of different terms used to describe the ways in which functions behave. In this section we explain some of these terms and illustrate their use.

Prerequisites

Before starting this Section you should ...

① understand what is meant by a function

② be able to graph simple functions

Learning Outcomes

After completing this Section you should be able to ...

✓ explain the distinction between a continuous and discontinuous function

✓ find the limits of simple functions

✓ explain what is meant by a periodic function

✓ explain what is meant by an odd function and an even function

1. Continuous and discontinuous functions and limits

Look at the graph shown in Figure 1a. The curve can be traced out from left to right without moving the pen from the paper. The function represented by this curve is said to be **continuous** at every point. If we try to trace out the curve in Figure 1b, the presence of a jump in the graph (at $x = x_1$) means that the pen must be lifted from the paper and moved in order to trace the graph. Such a function is said to be **discontinuous** at the point where the jump occurs. The jumps are known as **discontinuities.**

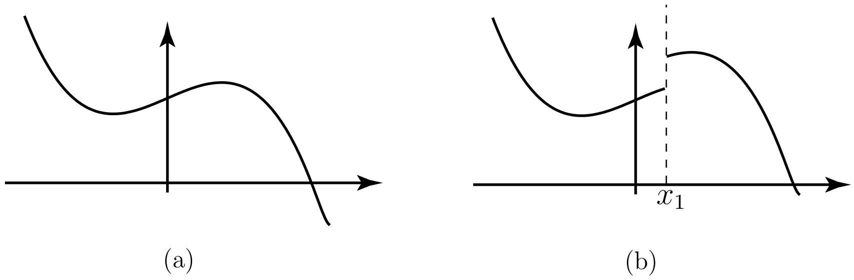


Figure 1. a) A continuous function. b) A discontinuous function.

Sketch a graph of a function which has two discontinuities.

Your solution

When defining a discontinuous function algebraically it is often necessary to give different function rules for different values of x. Conside, for example, the function defined as:

$$f(x) = \begin{cases} 3 & x < 0 \\ x^2 & x \geq 0 \end{cases}$$

Notice that there is one rule for when x is less than 0 and another rule for when x is greater than or equal to 0.

A graph of this function is shown below.

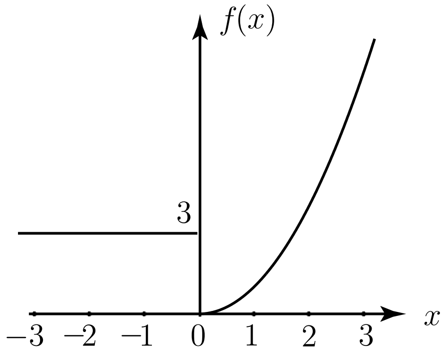


Figure 2 An example of a discontinuous function

Suppose we ask 'to what value does y approach as x approaches 0?'. From the graph we see that as x gets nearer and nearer to 0, the value of y gets nearer to 0, if we approach from the right-hand side. We write this formally as

$$\lim_{x\to 0^+} f(x) = 0$$

and say 'the limit of $f(x)$ as x tends to 0 from above is 0.'

On the other hand if x gets closer to zero, from the left-hand side, the value of y remains at 3. In this case we write

$$\lim_{x\to 0^-} f(x) = 3$$

and say 'the limit of $f(x)$ as x tends to 0 from below is 3.'

In this example the right-hand limit and the left-hand limit are not equal, and this is indicative of the fact that the function is discontinuous.
In general a function is continuous at a point $x = a$ if the left-hand and right-hand limits are the same there and are finite, and if both of these are equal to the value of the function at that point. That is

Key Point

A function $f(x)$ is continuous at $x = a$ if and only if:

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$$

If the right-hand and left-hand limits are the same, we can simply describe this common limit as $\lim_{x\to a} f(x)$. If the limits are not the same we say the limit of the function does not exist at $x = a$.

Exercises

1. Explain the distinction between a continuous and a discontinous function. Draw a graph showing an example of each type of function.
2. Study graphs of the functions $y = x^2$ and $y = -x^2$. Are these continuous functions ?
3. Study graphs of $y = 3x - 2$ and $y = -7x + 1$. Are these continuous functions?
4. Draw a graph of the function

$$f(x) = \begin{cases} 2x + 1 & x < 3 \\ 5 & x = 3 \\ 6 & x > 3 \end{cases}$$

Find
a) $\lim_{x\to 0^+} f(x)$, b) $\lim_{x\to 0^-} f(x)$, c) $\lim_{x\to 0} f(x)$, d) $\lim_{x\to 3^+} f(x)$,
e) $\lim_{x\to 3^-} f(x)$, f) $\lim_{x\to 3} f(x)$,

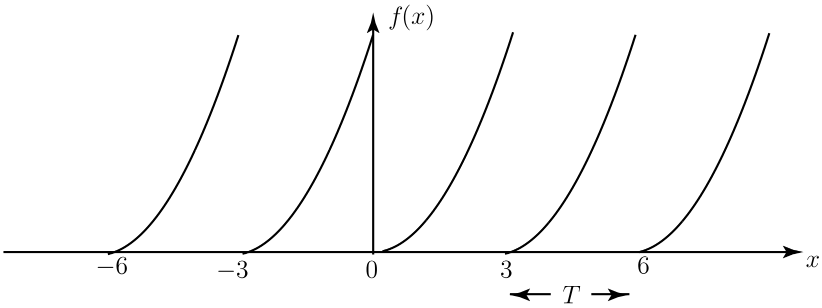
Answers 2. Yes. 3. Yes. 4. a) 1, b) 1, c) 1, d) 6, e) 7, f) limit does not exist.

2. Periodic functions

Any function that has a definite pattern repeated at regular intervals is said to be periodic. The interval over which the repetition takes place is called the **period** of the function, and is usually given the symbol T. The period of a periodic function is usually obvious from its graph.

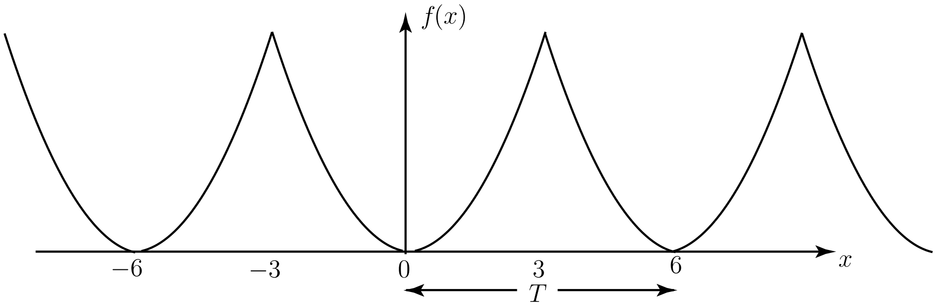
Example

The following figure shows a graph of a periodic function with period $T = 3$. This function has discontinuities at values of x which are divisible by 3.



Example

The following figure shows a graph of a periodic function with period $T = 6$. This function has no discontinuities.



If a function is a periodic function with period T then, for any value of the independent variable x, the value of $f(x+T)$ is the **same** as the value of $f(x)$.

Key Point

A function $f(x)$ is **periodic** if we can find a number T such that

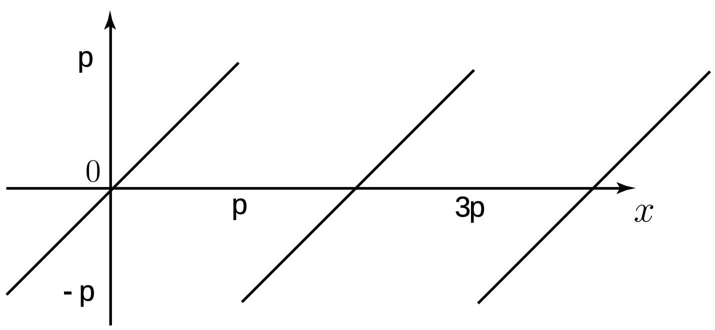
$$f(x+T) = f(x) \qquad \text{for all values of } x$$

Often a periodic function will be defined by simply specifying the period of the function and by stating the rule for the function within one period. This information alone is sufficient to draw the graph for all values of the independent variable.

Example

In the following figure we graph the periodic function defined by

$$f(x) = x, \quad -\pi < x < \pi, \qquad \text{period } T = 2\pi$$



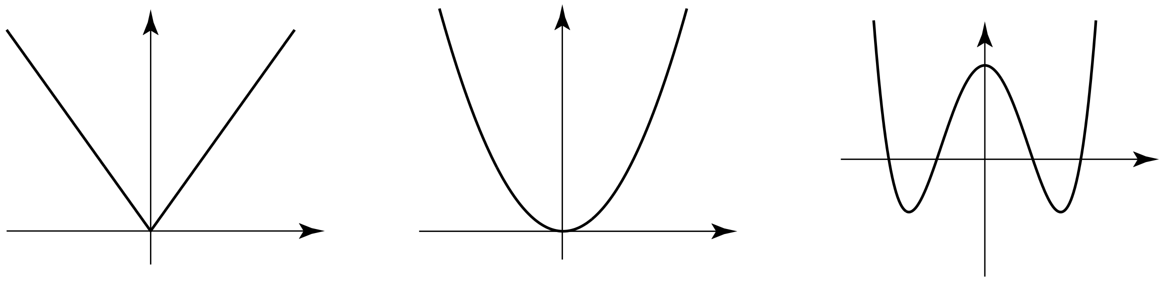
Exercises

1. Explain what is meant by a periodic function.
2. Sketch a graph of a periodic function which has no discontinuities.
3. Sketch a graph of a periodic function which has discontinuities.
4. A periodic function has period 0.01 seconds. How many times will the pattern in the graph repeat over an interval of 10 seconds ?

Answers 4. 1000.

3. Odd and even functions

The following figure shows graphs of several functions. They share a common property. Study the graphs and comment on any symmetry.



The graphs are all symmetrical about the y axis.

Any function which is symmetrical about the y axis, i.e. where the graph of the right hand part is the mirror image of that on the left, is said to be an **even** function. Even functions have the following property:

Key Point

An even function is such that $f(-x) = f(x)$ for all values of x.

This is saying that the function value at a negative value of x is the same as the function value at the corresponding positive value of x.

Example Show algebraically that $f(x) = x^4 + 5$ is an even function.

Solution

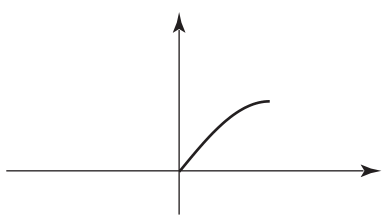
We must show that $f(-x) = f(x)$.

$$f(-x) = (-x)^4 + 5 = x^4 + 5$$

Hence $f(-x) = f(x)$ and so the function is even. Check for yourself that $f(-3) = f(3)$.

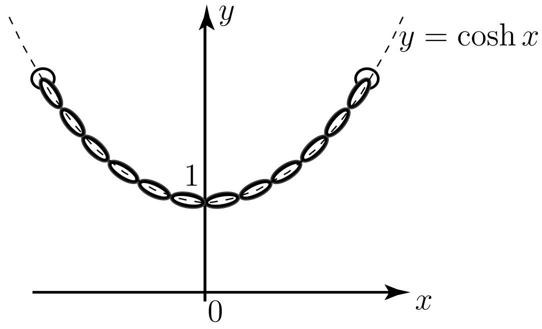
Extend the graph in the solution box in order to produce a graph of an even function.

Your solution



Extend this graph to produce an even function

The figure below represents a heavy cable hanging under gravity from two points at the same height. Such a curve (shown as a dashed line), known as a **catenary**, is described by a mathematical function known as a hyperbolic cosine, $f(x) = \cosh x$.



The catenary

a) Comment upon any symmetry.
b) Is this function one-to-one or many-to-one ?
c) Is this a continuous or discontinuous function ?
d) State $\lim_{x\to 0} \cosh x$.

Your solution

a) function is even, b) many-to-one, c) continuous, d) 1

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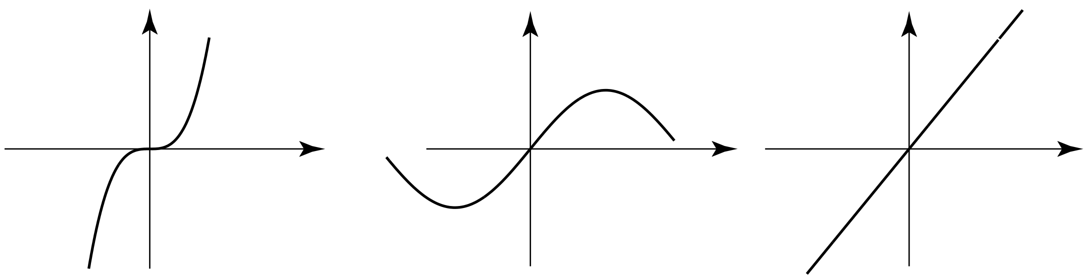
The following figure shows graphs of several functions. They share a common property. Study the graphs and comment on any symmetry.



Your solution

There is rotational symmetry about the origin. That is, each curve, when rotated through $180°$, transforms into itself.

Any function which possesses such symmetry, that is the graph of the right can be obtained by rotating the curve on the left through $180°$ about the origin, is said to be an **odd** function. Odd functions have the following property:

Key Point

An odd function is such that $f(-x) = -f(x)$ for all values of x.

Example Show that the function $f(x) = x^3 + 4x$ is odd.

Solution

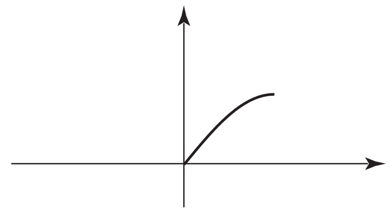
We must show that $f(-x) = -f(x)$.

$$\begin{aligned} f(-x) &= (-x)^3 + 4(-x) \\ &= -x^3 - 4x \\ &= -(x^3 + 4x) \\ &= -f(x) \end{aligned}$$

and so this function is odd. Check for yourself that $f(-2) = -f(2)$.

Extend the graph in the solution box in order to produce a graph of an odd function.

Your solution



Extend this graph to produce an odd function

Note that some functions are neither odd nor even; for example $f(x) = x^3 + x^2$ is neither even nor odd. The reader should confirm (with simple examples) that, when referring to functions, 'odd' and 'even' have the following properties:

$$\text{odd} + \text{odd} = \text{odd} \qquad \text{even} + \text{even} = \text{even} \qquad \text{odd} + \text{even} = \text{neither}$$
$$\text{odd} \times \text{odd} = \text{even} \qquad \text{even} \times \text{even} = \text{even} \qquad \text{odd} \times \text{even} = \text{odd}$$

Exercises

1. Classify the following functions as odd, even or neither. If necessary sketch a graph to help you decide.
a) $f(x) = 6$, b) $f(x) = x^2$, c) $f(x) = 2x + 1$, d) $f(x) = x$, e) $f(x) = 2x$

Answers 1 a) even, b) even, c) neither, d) odd, e) odd