The straight line

2.5

Introduction

Probably the most important function and graph that you will use are those associated with the straight line. A large number of relationships between engineering variables can be described using a straight line or **linear** graph. Even when this is not strictly the case it is often possible to approximate a relationship by a straight line. In this section we study the equation of a straight line, its properties and graph.

Prerequisites

Before starting this Section you should . . .

① understand what is meant by a function

② be able to graph simple functions

Learning Outcomes

After completing this Section you should be able to . . .

✓ recognise the equation of a straight line

✓ explain the significance of a and b in the equation of a line $f(x) = ax + b$

✓ find the gradient of a line given two points on the line

✓ find the equation of a line joining two points

✓ find the distance between two points on a line

1. Linear functions

Any function of the form $y = f(x) = ax + b$ where a and b are constants is called a **linear function.** The constant a is called the **coefficient of** x, and b is referred to as the **constant term.**

Key Point

All linear functions can be written in the form:

$$f(x) = ax + b$$

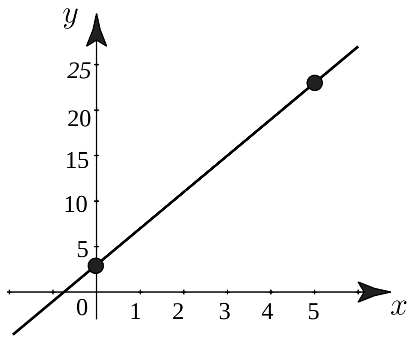
where a and b are constants.

For example, $f(x) = 3x + 2$, $g(x) = \frac{1}{2}x - 7$, $h(x) = -3x + \frac{2}{3}$ and $k(x) = 2x$ are all linear functions.
The graph of a linear function is always a straight line. Such a graph can be plotted by finding just two distinct points and joining these with a straight line.

Example Plot the graph of the linear function $y = f(x) = 4x + 3$.

Solution

We start by finding two points. For example if we choose $x = 0$, then $y = f(0) = 3$, i.e. the first point has coordinates $(0, 3)$. Secondly, suppose we choose $x = 5$, then $y = f(5) = 23$. The second point is $(5, 23)$. These two points are then plotted and then joined by a straight line as shown in the following figure.



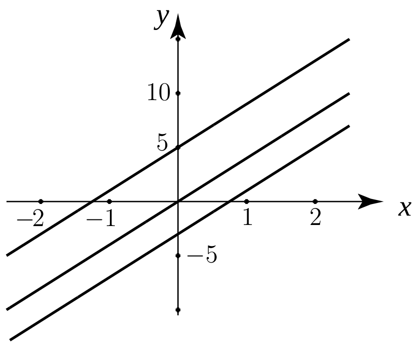
Example

Plot graphs of the three linear functions $y = 4x - 3$, $y = 4x$, and $y = 4x + 5$, for $-2 \leq x \leq 2$.

Solution

For each function it is necessary to find two points on the line.

For $y = 4x - 3$, suppose for the first point we choose $x = 0$, so that $y = -3$. For the second point, let $x = 2$ so that $y = 5$. So, the points $(0, -3)$ and $(2, 5)$ can be plotted and joined. This is shown in the following figure.



For $y = 4x$ we find the points $(0, 0)$ and $(2, 8)$. Similarly for $y = 4x + 5$ we find points $(0, 5)$ and $(2, 13)$. The corresponding lines are also shown in the figure.

Refer to Example 2. Comment upon the effect of changing the value of the constant term of the linear function.

Your solution

As the constant term is varied, the line moves up or down the page always remaining parallel to itself.

The value of the constant term is also known as the **vertical or y-axis intercept** because this is the value of y where the line cuts the y axis.

State the vertical intercept of each of the following lines:
a) $y = 3x + 3$, b) $y = \frac{1}{2}x - \frac{1}{3}$, c) $y = 1 - 3x$, d) $y = -5x$.

Your solution

In each case identify the constant term.

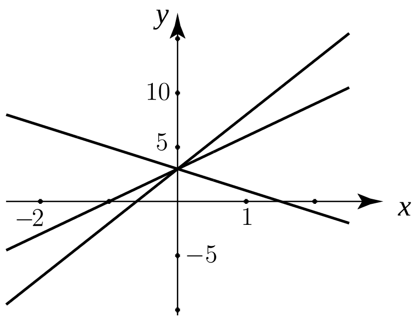
a) b) c) d)

a) 3, b) $-\frac{1}{3}$, c) 1, d) 0

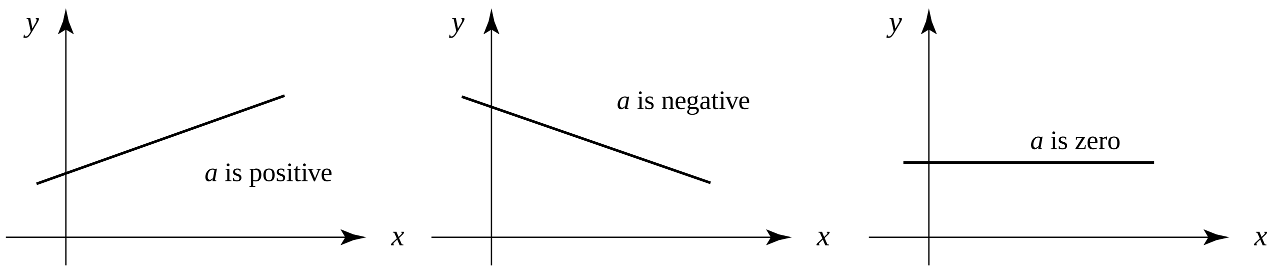
Example Plot graphs of the lines $y = 3x + 3$, $y = 5x + 3$ and $y = -2x + 3$.

Solution

Note that all three lines have the same constant term, that is 3. So all three lines pass through $(0, 3)$, the vertical intercept. A further point has been calculated for each of the lines and their graphs are shown in the following figure



Note from the graphs in Example 3 that as the coefficient of x is changed the gradient of the graph changes. The coefficient of x gives the **gradient** or **slope** of the line. In general, for the line $y = ax + b$ a positive value of a produces a graph which slopes upwards from left to right. A negative value of a produces a graph which slopes downwards from left to right. If a is zero the line is horizontal, that is its gradient is zero. These properties are summarised in the next figure.



The gradient of a line $y = ax + b$ depends upon the value of a.

Key Point

In the linear function $f(x) = ax + b$, a is the gradient and b is the vertical intercept.

State the gradients of the following lines:
a) $y = 7x + 2$ b) $y = -\frac{1}{3}x + 4$ c) $y = \frac{x+2}{3}$

Your solution

In each case the coefficient of x must be examined.

a) b) c)

a) 7, b) $-1/3$, c) $1/3$

Which of the following lines has the steepest gradient ?
a) $y = \frac{17x+4}{5}$, b) $y = 9x - 2$, c) $y = \frac{1}{3}x + 4$.

Your solution

b)

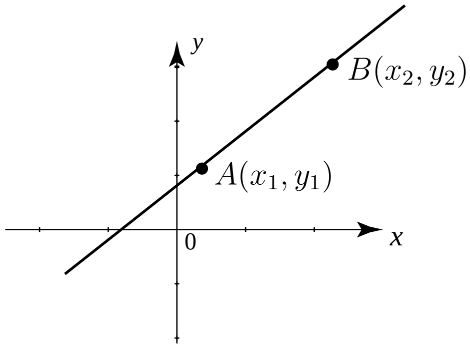
Exercises

1. State the general form of the equation of a straight line explaining the role of each of the terms in your answer.
2. State which of the following functions will have straight line graphs.
a) $f(x) = 3x - 3$, b) $f(x) = x^{1/2}$, c) $f(x) = \frac{1}{x}$, d) $f(x) = 13$, e) $f(x) = -2 - x$.
3. For each of the following, identify the gradient and vertical intercept.
a) $f(x) = 2x + 1$, b) $f(x) = 3$, c) $f(x) = -2x$, d) $f(x) = -7 - 17x$, e) $f(x) = mx + c$.

Answers 1. e.g. $y = ax + b$. x is the independent variable, y is the dependent variable, a is the gradient and b is the vertical intercept.
2. a), d) and e) will have straight line graphs.
3. a) gradient = 2, vertical intercept =1, b) 0,3, c) -2,0, d) -17, -7, e) m, c.

2. Finding the gradient of a line given two points on the line

A common requirement is to find the gradient of a line when we know the coordinates of two points on it. Suppose the two points are $A(x_1, y_1)$, $B(x_2, y_2)$ as shown in the following figure.



The gradient of the line joining A and B can be calculated from the following formula.

Key Point

The gradient of the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

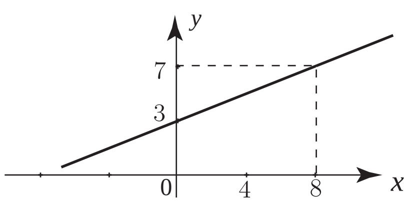
$$\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example Find the gradient of the line joining the points $A(0, 3)$ and $B(4, 5)$.

Solution

We calculate the gradient as follows:

$$\begin{aligned} \text{gradient} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{5 - 3}{4 - 0} \\ &= \frac{1}{2} \end{aligned}$$



Thus the gradient of the line is $\frac{1}{2}$. Graphically, this means that when x increases by 1, the value of y increases by $\frac{1}{2}$.

Find the gradient of the line joining the points $A(-1, 4)$ and $B(2, 1)$.

Your solution

gradient $= \dfrac{y_2 - y_1}{x_2 - x_1} =$

$\frac{1-4}{2-(-1)} = -1$

Thus the gradient of the line is -1. Graphically, this means that when x increases by 1, the value of y decreases by 1.

Exercises

1. Calculate the gradient of the line joining $(1, 0)$ and $(15, -3)$.
2. Calculate the gradient of the line joining $(10, -3)$ and $(15, -3)$.

Answers 1. $-3/14$. 2. 0

3. Finding the equation of a line joining two points

The equation of the line passing through the points with coordinates $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by the following formula.

Key Point

The line passing through points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \qquad \text{or, equivalently} \qquad y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)x + \frac{x_2y_1 - y_2x_1}{x_2 - x_1}$$

Find the equation of the line passing through $A(-7, 11)$ and $B(1, 3)$.

Your solution

Apply the formula: $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$ $\qquad \dfrac{y - \quad}{\quad - \quad} = \dfrac{x - \quad}{\quad - \quad}$

$\frac{y-11}{3-11} = \frac{x+7}{1+7}$. Simplify this to obtain the required equation. $y = 4 - x$

Exercises

1. Find the equation of the line joining $(1, 5)$ and $(-9, 2)$.
2. Find the gradient and vertical intercept of the line joining $(8, 1)$ and $(-2, -3)$.

Answers 1. $y = \frac{3}{10}x + \frac{47}{10}$. 2. 0.4, -2.2.

4. Finding the distance between two points

Referring again to the figure of Section 2, the distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given using Pythagoras' theorem by the following formula.

Key Point

The distance between points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Find the distance between $A(-7, 11)$ and $B(1, 3)$

Your solution

Apply the formula: distance $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} =$

$\sqrt{128}$

Exercises

1. Find the distance between the points $(4, 5)$ and $(-17, 1)$.

Answers 1. $\sqrt{457}$