# The Circle

2.6

## Introduction

A circle is one of the most familiar geometrical figures. In this brief Section we discuss the basic coordinate geometry of a circle - in particular the basic equation representing a circle in terms of its centre and radius.

## Prerequisites

Before starting this Section you should ...

① understand what is meant by a function and use functional notation

② be able to plot graphs of functions

## Learning Outcomes

After completing this Section you should be able to ...

✓ write the equation of any given circle

✓ obtain the centre and radius of a circle from its equation

## 1. Equations for Circles in the $0xy$ plane

The obvious characteristic of a circle is that every point on its circumference is the **same** distance from the **centre**. This fixed distance is called the **radius** of the circle and is generally denoted by $R$ or $r$ or $a$.

In coordinate geometry terms suppose $(x, y)$ denotes the coordinates of a point e.g. (4,2) means $x = 4,\ y = 2$, $(-1, 1)$ means $x = -1,\ y = 1$ and so on. See Figure 1.

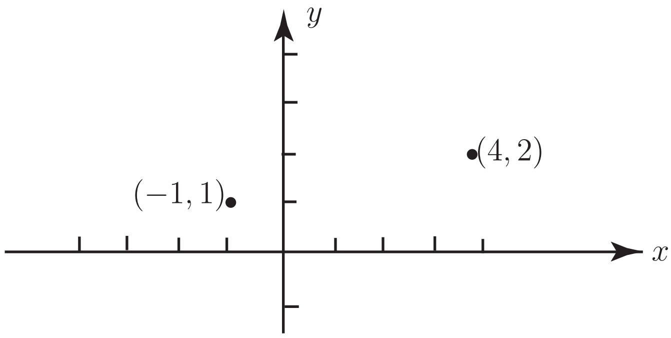


Figure 1

Write down the distances $d_1$ and $d_2$ from the origin of the points with coordinates (4,2) and $(-1, 1)$ respectively. Generalise your results to obtain the distance $d$ from the origin of **any** arbitrary point with coordinates $(x, y)$.

**Your solution**

Using Pythagoras' Theorem:

(a) $d_1 = \sqrt{4^2 + 2^2} = \sqrt{20}$ is the distance between the origin (0,0) and the point (4,2).

(b) $d_2 = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$ is the distance between the origin and $(-1, 1)$.

(c) $d = \sqrt{x^2 + y^2}$ is the distance from the origin to an arbitrary point $(x, y)$.

Note that the positive square root is taken in each case.

## Circles with centre at the origin

Suppose $(x, y)$ is any point $P$ on a circle of radius $R$ whose centre is at the origin. See Figure 2.

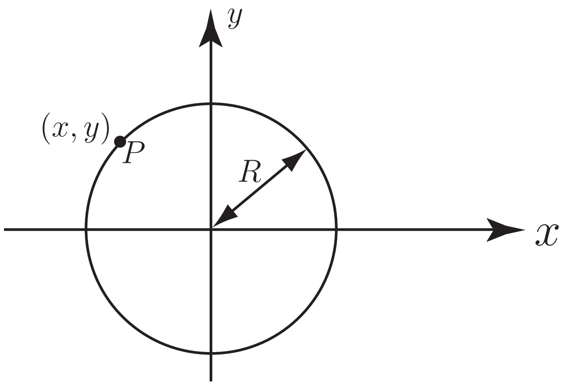


Figure 2

Bearing in mind the final result of the previous guided exercise, write down an equation relating $x$, $y$ and $R$.

**Your solution**

Since $\sqrt{x^2 + y^2}$ is distance of any point $(x, y)$ from the origin, then for any point $P$ on the above circle.

$$\sqrt{x^2 + y^2} = R \qquad \text{or} \qquad x^2 + y^2 = R^2$$

The whole point is that as the point $P$ in Figure 2 moves around the circle its $x$ and $y$ coordinates change. However $P$ will remain at the same distance $R$ from the origin by the very definition of a circle.
Hence we say that

$$\sqrt{x^2 + y^2} = R \quad \text{or, more usually,}$$

$$x^2 + y^2 = R^2 \qquad (1)$$

is **the equation** of the circle radius $R$ centre at the origin. What this means is that if a point $(x, y)$ satisfies (1) then it lies on the circumference of the circle radius $R$. If $(x, y)$ does not satisfy (1) then it does not lie on that circumference.
Note carefully that the right hand sides of the circle equation (1) is the **square** of the radius.

Consider the circle centre at the origin and of radius 5.

(a) Write down the equation of this circle.

(b) Find which of the following points lie on the circumference of this circle.

$$(5,0)\ \ (0,-5)\ \ (4,3)\ \ (-3,4)\ \ (2,\sqrt{21})\ \ (-2\sqrt{6},1)\ \ (1,4)\ \ (4,-4)$$

(c) For those points in (ii) which do not lie on the circle deduce whether they lie inside or outside the circle.

**Your solution**

(a) $x^2+y^2=5^2=25$ is the equation of the circle.

(b) For each point $(x,y)$ we calculate $x^2+y^2$. If this equals 25 the point lies on the circle.

(c)

| $x,y$ | $x^2+y^2$ | conclusion |
|---|---|---|
| (5,0) | 25 | on circle |
| $(0,-5)$ | 25 | on circle |
| (4,3) | 25 | on circle |
| $(-3,-4)$ | 25 | on circle |
| $(2,\sqrt{21})$ | 25 | on circle |
| $(-2\sqrt{6},1)$ | 25 | on circle |
| (1,4) | 17 | inside circle |
| $(4,-4)$ | 32 | outside circle |

Figure 3 demonstrates some of the results of the previous example.

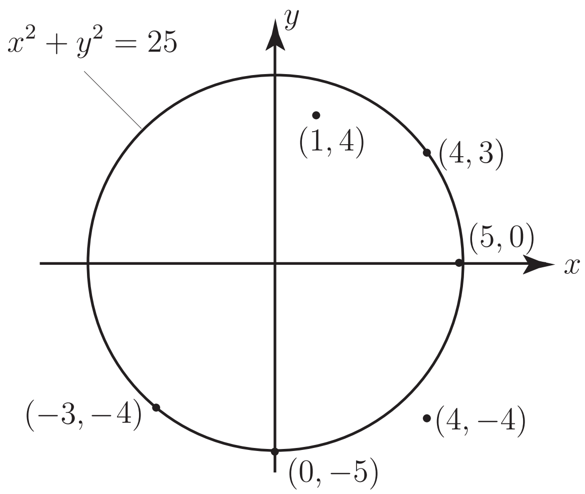


Figure 3

Note that the circle centre at the origin and of radius 1 has a special name – the **unit circle.**

(a) Calculate the distance between the points $P_1(-1, 1)$ and $P_2(4, 2)$. Refer back to Figure 1.

(b) Generalise your result to obtain the distance between any two points whose coordinates are $(x_1, y_1)$ and $(x_2, y_2)$.

**Your solution**

(i)

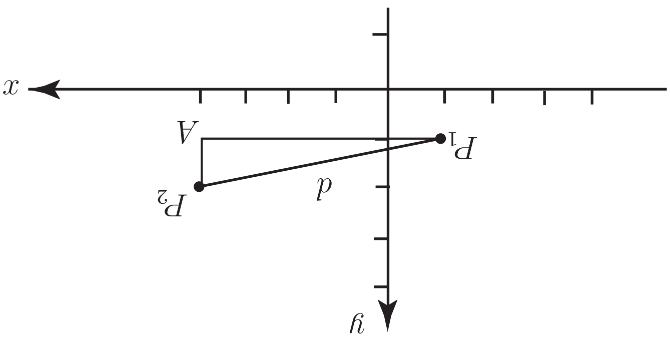


Figure 4

Using Pythagoras' Theorem the distance between the two given points is

$$d = \sqrt{(P_1A)^2 + (AP_2)^2}$$

where $P_1A = 4 - (-1) = 5, \quad AP_2 = 2 - 1 = 1 \quad \therefore \quad d = \sqrt{5^2 + 1^2} = \sqrt{26}$

(ii)

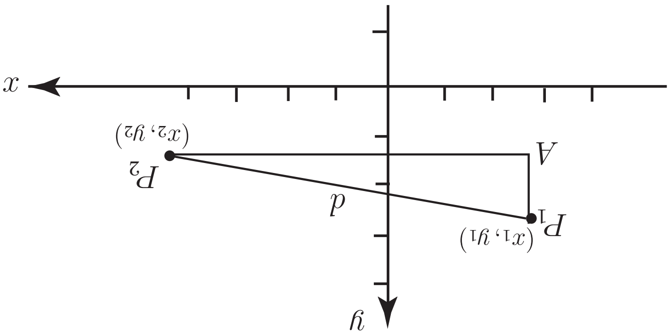


Figure 5

Between the arbitrary points $P_1$ and $P_2$ the distance is

$$d = \sqrt{(AP_2)^2 + (P_1A)^2}$$

where $AP_2 = x_2 - x_1$, $P_1A = y_1 - y_2 = -(y_2 - y_1)$

so $\quad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Remembering the result of part (ii) of this example we now consider a circle centre at the point $C(x_0; y_0)$ and of radius $R$. Suppose $P$ is an arbitrary point on this circle which has co-ordinates $(x, y)$

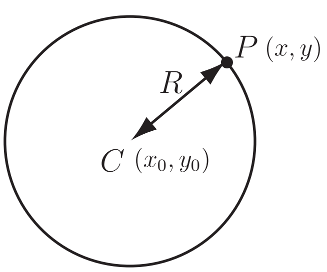


Figure 6

Clearly $R = CP = \sqrt{(x - x_0)^2 + (y - y_0)^2}$
Hence, squaring both sides,

$$(x - x_0)^2 + (y - y_0)^2 = R^2 \qquad (2)$$

which is said to be the equation of the circle centre $(x_0, y_0)$ radius $R$.

Note that if $x_0 = y_0 = 0$ (i.e. circle centre is at origin) then (2) reduces to (1) so the latter is simply a special case.

The interpretation of (2) is similar to that of (1): any point $(x, y)$ satisfying (2) lies on the circumference of the circle.

**Example**

The equation

$$(x - 3)^2 + (y - 4)^2 = 4$$

represents a circle of radius 2 (the positive square root of 4) and has centre $C$ (3,4).

N.B. There is no need to expand the terms on the left-hand side here. The given form reveals quite plainly the radius and centre of the circle.

Write down the equations of each of the following circles. The centre $C$ and radius $R$ are given:

(a) $C(0, 2),\ R = 2$

(b) $C(-2, 0),\ R = 3$

(c) $C(-3, 4),\ R = 5$

(d) $C(1, 1),\ R = \sqrt{3}$

**Your solution**

(a) $x_0 = 0$, $y_0 = 2$, $R^2 = 4$ so by (2)

$$x^2 + (y-2)^2 = 4$$

is the required equation.

(b) $x_0 = -2$, $y_0 = 0$, $R^2 = 9$

$$\therefore \quad (x+2)^2 + y^2 = 9$$

(c) $(x+3)^2 + (y-4)^2 = 25$

(d) $(x-1)^2 + (y-1)^2 = 3$

Again we emphasize that the right hand side of each of these equations is the square of the radius.

Write down the equations of each of the circles shown

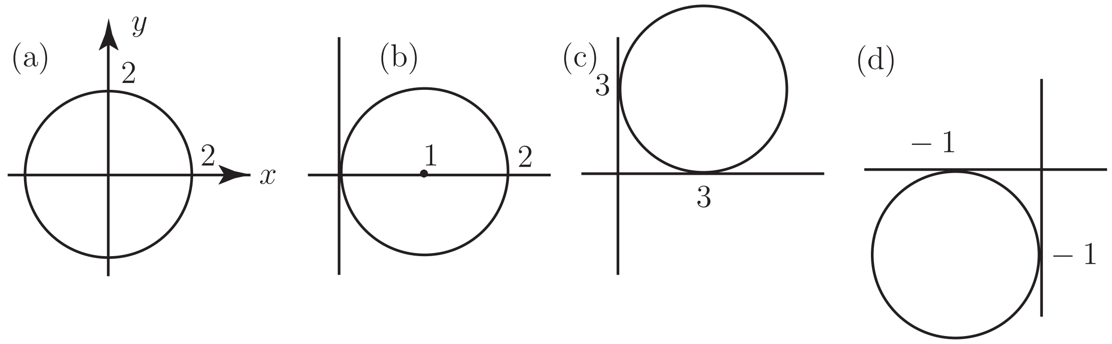


Figure 7

**Your solution**

(a) $x^2 + y^2 = 4$ (centre origin, radius 2)

(b) $(x-1)^2 + y^2 = 1$ (centre (1,0) radius 1)

(c) $(x-3)^2 + (y-3)^2 = 0$ (centre (3,3) radius 3)

(d) $(x+1)^2 + (y+1)^2 = 1$ (centre $(-1,-1)$ radius 1)

Consider again the equation of the circle, centre (3,4) of radius 2:

$$(x-3)^2 + (y-4)^2 = 4 \tag{3}$$

In this form of the equation the centre and radius of the circle can be clearly identified and, as we said, there is no advantage in squaring out.
However, if we did square out the equation would become

$$x^2 - 6x + 9 + y^2 - 8x + 16 = 4 \quad \text{or} \quad x^2 - 6x + y^2 - 8x + 21 = 0 \tag{4}$$

Equation (4) is of course a valid equation for this circle but, we cannot immediately obtain the centre and radius from it.

For the case of the general circle of radius $R$

$$(x - x_0)^2 + (y - y_0)^2 = R^2$$

expand out the square terms and simplify.

**Your solution**

We obtain

$$x^2 - 2x_0x + x_0^2 + y^2 - 2y_0y + y_0^2 - R^2 = 0$$

or

$$x^2 + y^2 - 2x_0x - 2y_0y + c = 0$$

where $c = x_0^2 + y_0^2 - R^2$.

It follows from the above exercise that any equation of the form

$$x^2 + y^2 - 2gx - 2fy + c = 0 \tag{5}$$

represents a circle with centre $(g, f)$ and a radius obtained by solving

$$c = g^2 + f^2 - R^2$$

for $R$.
Thus

$$R = \sqrt{g^2 + f^2 - c} \qquad (6)$$

There is no reason to remember equation (6). In any specific problem the technique of **completion of square** can be used to turn an equation of the form (5) into the form (2) and hence obtain the centre and radius of the circle.

NB. The key point about (5) is that the coefficients of the term $x^2$ and $y^2$ are the **same**, i.e. 1. An equation with the coefficient of $x^2$ and $y^2$ identical with value $k \neq 1$ could be converted into the form (5) by division of the whole equation by $k$.

If

$$x^2 + y^2 - 2x + 10y + 16 = 0$$

obtain the centre and radius of the circle that this equation represents. Begin by completing the square separately on the $x-$terms and the $y-$terms.

**Your solution**

$$\begin{aligned} x^2 - 2x &= (x-1)^2 - 1 \\ y^2 + 10y &= (y+5)^2 - 25 \end{aligned}$$

Now complete the problem.

**Your solution**

The original equation

$$x^2 + y^2 - 2x + 10y + 16 = 0$$

becomes

$$(x-1)^2 - 1 + (y+5)^2 - 25 + 16 = 0$$

$$\therefore \quad (x-1)^2 + (y+5)^2 = 10$$

which represents a circle with centre $(1, -5)$ and radius $\sqrt{10}$.

## Circles and functions

Let us return to the equation of the unit circle

$$x^2 + y^2 = 1$$

Solving for $y$ we obtain

$$y = \pm\sqrt{1 - x^2}.$$

This equation does not represent a function because of the two possible square roots which imply that for any value of $x$ there are *two* values of $y$. (You will recall from earlier in this Workbook that a function requires only *one* value of the dependent variable $y$ corresponding to each value of the independent variable $x$.)
However two functions can be obtained in this case:

$$y = y_1 = +\sqrt{1 - x^2} \qquad y = y_2 = -\sqrt{1 - x^2}$$

whose graphs are the semicircles shown.

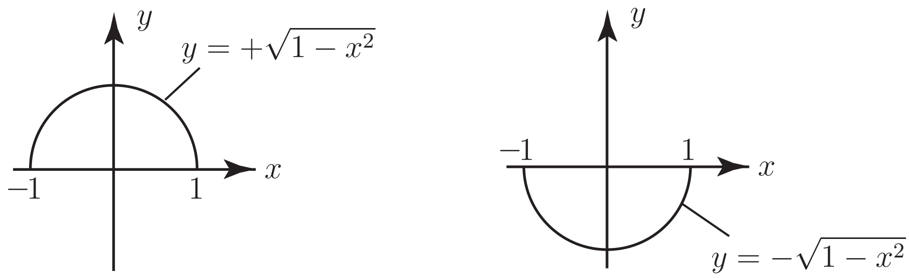


Figure 8

## Annuli between circles

Equations in $x$ and $y$, such as (1) and (2) for circles, define **curves** in the $Oxy$ plane. However, inequalities are necessary to define **regions**. For example, the inequality

$$x^2 + y^2 < 1$$

is satisfied by, all points **inside** the unit circle - for example $(0,0)$ $(0, \frac{1}{2})$ $(\frac{1}{4}, 0)$ $(\frac{1}{2}, \frac{1}{2})$.
Similarly

$$x^2 + y^2 > 1$$

is satisfied by all points outside that circle such as (1,1).

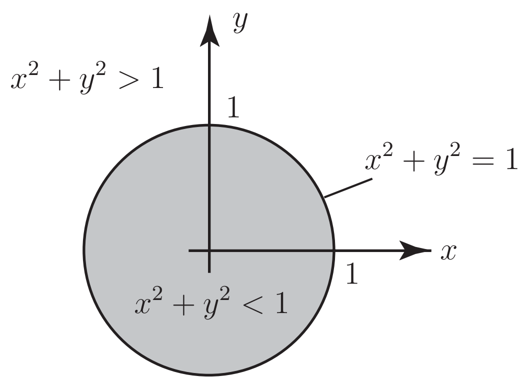

Figure 9

Sketch the regions in the $Oxy$ plane defined by

(a) $(x-1)^2+y^2<1$

(b) $(x-1)^2+y^2>1$

**Your solution**

The equality

$$(x-1)^2+y^2=1 \qquad (7)$$

is satisfied by any point on the circumference of the circle centre (1,0) radius 1. Then, remembering that $(x-1)^2+y^2$ is the square of the distance between any point $(x,y)$ and (1,0), it follows that

(a) $(x-1)^2+y^2<1$ is satisfied by any point inside this circle (Region (A) on the following diagram.)

(b) $(x-1)^2+y^2>1$ defines the region exterior to the circle since this inequality is satisfied by every point outside. (Region (B) on the diagram.)

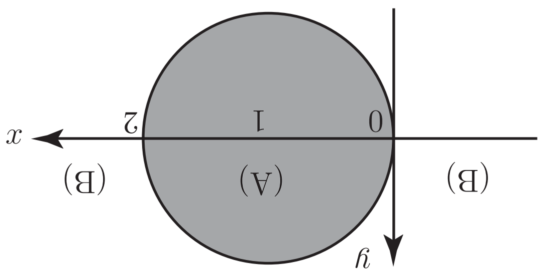


Figure 10

The region between two circles with the same centre (concentric circles) is called an annulus or annular region. An annulus is defined by **two** inequalities.

For example the inequality

$$x^2 + y^2 > 1 \qquad (8)$$

defines, as we saw, the region outside the unit circle.

The inequality

$$x^2 + y^2 < 4 \qquad (9)$$

defines the region inside the circle centre origin radius 2.

Hence points $(x, y)$ which satisfy **both** the inequalities (8) and (9) lie in the annulus between the two circles. The inequalities (8) and (9) are combined by writing

$$1 < x^2 + y^2 < 4$$

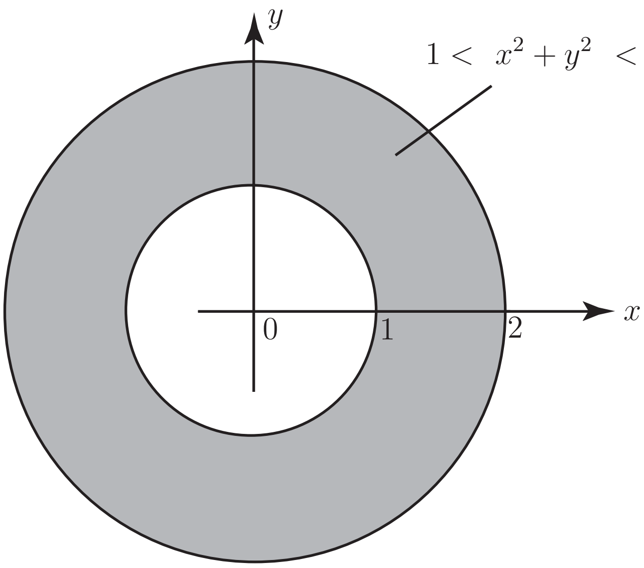


Figure 11

Sketch the annulus defined by the inequalities

$$1 < (x - 1)^2 + y^2 < 9 \qquad (10)$$

(Refer back to the previous but one Guided Exercise if necessary.)

The quantity $(x-1)^2+y^2$ is the square of the distance of a point $(x,y)$ from the point (1,0). Hence, as we saw earlier, the left-hand inequality

$$1 < (x-1)^2 + y^2$$

or, equivalently,

$$(x-1)^2 + y^2 > 1$$

is the region exterior to the circle $C_1$ centre (1,0) radius 1. Similarly the right had inequality

$$(x-1)^2 + y^2 < 9$$

defines the interior of the circle $C_2$ centre (1,0) radius 3. Hence the inequality (10) holds for any point in the annulus between $C_1$ and $C_2$.

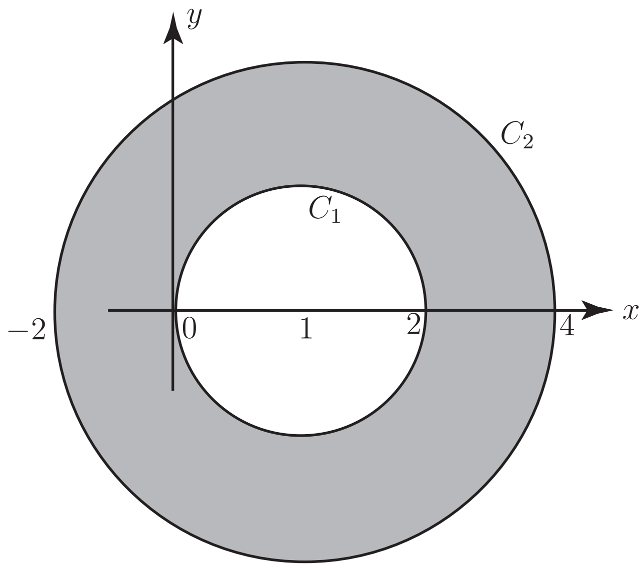


Figure 12

**Your solution**

## Exercises

1. Write down the radius and the coordinates of the centre of the circle for each of the following equations

   (a) $x^2 + y^2 = 16$

   (b) $(x-4)^2 + (y-3)^2 = 12$

   (c) $(x+3)^2 + (y-1)^2 = 25$

   (d) $x^2 + (y+1)^2 - 4 = 0$

   (e) $(x+6)^2 + y^2 - 36 = 0$

2. Obtain in each case the equation of the given circle

   (a) centre $C$ $(0,0)$ radius 7

   (b) centre $C$ $(0,2)$ radius 2

   (c) centre $C$ $(4,-4)$ radius 4

   (d) centre $C$ $(-2,-2)$ radius 4

   (e) centre $C$ $(-6,0)$ radius 5

3. Obtain the radius and the coordinates of the centre for each of the following circles

   (a) $x^2 + y^2 - 10x + 12y = 0$

   (b) $x^2 + y^2 + 2x - 4y = 11$

   (c) $x^2 + y^2 - 6x - 16 = 0$

4. Describe the regions defined by each of these inequalities

   (a) $x^2 + y^2 > 4$

   (b) $x^2 + y^2 < 16$

   (c) the inequalities in (i) and (ii) together

5. Write an inequality that describes the points that lie outside the circle of radius 4 with centre $(-4,2)$.

6. Write an inequality that describes the points that lie inside the circle of radius $\sqrt{6}$ with centre $(-2,-1)$.

7. Obtain the equation of the circle which has centre $(3,4)$ and which passes through the point $(0,5)$.

8. Show that if $A(x_1,y_1)$ and $B(x_2,y_2)$ are at opposite ends of a diameter of a circle then the equation of the circle is $\quad (x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$.

   (Hint: if $P$ is any point on the circle obtain the slopes of the lines $AP$ and $BP$ and recall that the angle in a semi-circle must be a right-angle.)

9. State the equation of the unique circle which **touches** the $x-$axis at the point (2,0) and which passes through the point $(-1,9)$.

**Answers**

1. (a) radius 4 centre $(0,0)$
   (b) radius $\sqrt{12}$ centre $(4,3)$
   (c) radius 5 centre $(-3,1)$
   (d) radius 2 centre $(0,-1)$
   (e) radius 6 centre $(-6,0)$

2. (a) $x^2+y^2=49$
   (b) $x^2+(y-2)^2=4$
   (c) $(x-4)^2+(y+4)^2=16$
   (d) $(x+2)^2+(y+2)^2=16$
   (e) $(x+6)^2+y^2=25$

3. (a) centre $(5,-6)$ radius $\sqrt{61}$
   (b) centre $(-1,2)$ radius 4
   (c) centre (3,0) radius 5

4. (a) the region outside the circumference of the circle centre the origin radius 2.
   (b) the region inside the circle centre the origin radius 4 (often referred to as a circular disc)
   (c) the annular ring between these two circles.

5. $(x+4)^2+(y-2)^2>16$

6. $(x+2)^2+(y+1)^2<6$

7. $(x-3)^2+(y-4)^2=10$

8. $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$.

9. $(x-2)^2+(y-5)^2=25$ (Note: since we are told the circle touches the $x-$axis at (2,0) the centre of the circle must be at the point $(2,y_0)$ where $y_0=R$).