Some common engineering functions

2.7

Introduction

This section provides a catalogue of some common functions often used in Science and Engineering. These include polynomials, rational functions, the modulus function and the unit step function. Important properties and definitions are stated. This section can be used as a reference when the need arises. There are, of course, other types of function which arise in engineering applications, such as trigonometric, exponential and logarithm functions. These others are dealt with in Workbooks 4 to 6.

Prerequisites

Before starting this Section you should . . .

① understand what is meant by a function and use functional notation

② be able to plot graphs of functions

Learning Outcomes

After completing this Section you should be able to . . .

✓ understand what is meant by a polynomial function, and a rational function

✓ be able to use and graph the modulus function

✓ be able to use and graph the unit step function

1. Polynomial Functions

A very important type of function is the **polynomial**. Polynomial functions are made up of multiples of non-negative whole number powers of a variable, such as $3x^2$, $-7x^3$ and so on. You are already familiar with many such functions. Other examples include:

$P_0(t) = 6$
$P_1(t) = 3t + 9$ (The linear function you have already met).
$P_2(x) = 3x^2 - x + 2$
$P_4(z) = 7z^4 + z^2 - 1$

where t, x and z are independent variables.

Note that fractional and negative powers of the independent variable are not allowed so that $f(x) = x^{-1}$ and $g(x) = x^{3/2}$ are not polynomials. The function $P_0(t) = 6$ is a polynomial - we can regard it as $6t^0$.

By convention a polynomial is written with the powers either increasing or decreasing. For example the polynomial

$$3x + 9x^2 - x^3 + 2$$

would be written as

$$-x^3 + 9x^2 + 3x + 2 \qquad \text{or} \qquad 2 + 3x + 9x^2 - x^3$$

In general we have the following definition:

Key Point

A **polynomial expression** has the form

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$$

where n is a non-negative integer, a_n, a_{n-1}, $\ldots, a_1$, a_0 are constants and x is a variable. A **polynomial function** $P(x)$ has the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$$

The **degree** of a polynomial or polynomial function is the value of the highest power. Referring to the examples listed above, polynomial P_2 has degree 2, because the term with the highest power is $3x^2$, P_4 has degree 4, P_1 has degree 1 and P_0 has degree 0. Polynomials with low degrees have special names given in Table 1.

Table 1

	degree	name
a	0	constant
$ax+b$	1	linear
ax^2+bx+c	2	quadratic
ax^3+bx^2+cx+d	3	cubic
$ax^4+bx^3+cx^2+dx+e$	4	quartic

Typical graphs of some polynomial functions are shown in Figure 1. In particular, observe that the graphs of the linear polynomials, P_1 and Q_2 are straight lines.

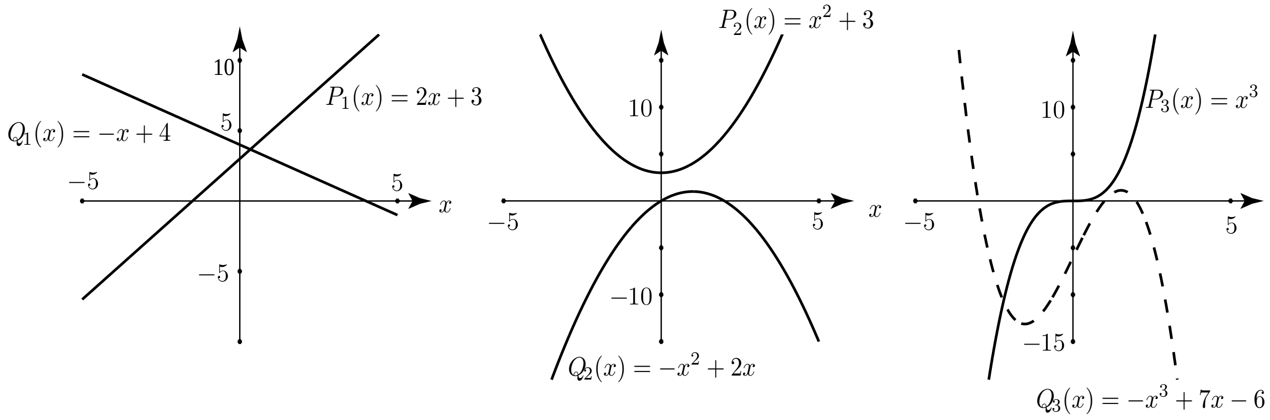


Figure 1. Graphs of some typical linear, quadratic and cubic polynomials

Which of the polynomial graphs in Figure 1 are odd and which are even? Are any periodic ?

Your solution

P_2 is even, P_3 is odd. None are periodic

State which of the following are polynomial functions. For those that are, give the degree and name. a) $f(x) = 6x^2+7x^3-2x^4$ b) $f(t) = t^3-3t^2+7$ c) $g(x) = \frac{1}{x^2}+\frac{3}{x}$ d) $f(x) = 16$ e) $g(x) = \frac{1}{6}$

Your solution

a) polynomial of degree 4 (quartic), b) polynomial of degree 3 (cubic), c) not a polynomial, d) polynomial of degree 0 (constant), e) polynomial of degree 0 (constant)

Exercises

1. Write down a polynomial of degree 3 with independent variable t.
2. Write down a function which is not a polynomial.
3. Explain why $y = 1 + x + x^{1/2}$ is not a polynomial.
4. State the degree of the following polynomials:

a) $P(t) = t^4 + 7$, b) $P(t) = -t^3 + 3$, c) $P(t) = 11$, d) $P(t) = t$

5. Write down a polynomial of degree 0 with independent variable z.
6. Referring to Figure 1, state which functions are one-to-one and which are many-to-one.

Answers 1. For example $f(t) = 1 + t + 3t^2 - t^3$.

2. For example $y = \frac{1}{x}$.

3. A term such as $x^{1/2}$, with a fractional index, is not allowed in a polynomial.

4. a) 4, b) 3, c) 0, d) 1. 5. $P(z) = 13$, for example. 6. P_1, Q_1 and P_3 are one-to-one. The rest are many-to-one.

2. Rational Functions

A rational function is formed by dividing one polynomial by another. Examples include

$$R_1(x) = \frac{x+6}{x^2+1}, \qquad R_2(t) = \frac{t^3-1}{2t+3}, \qquad R_3(z) = \frac{2z^2+z-1}{z^2+z-2}$$

For convenience we have labelled these rational functions R_1, R_2 and R_3.

Key Point

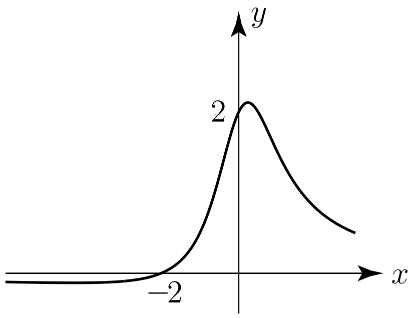
A **rational function** has the form

$$R(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomial functions, P is called the **numerator** and Q is called the **denominator**.

The graphs of rational functions can take a variety of different forms and can be difficult to plot by hand. Use of a graphics calculator or computer software can help. If you have access to a plotting package or calculator it would be useful to obtain graphs of these functions for yourself. The next three Examples allow you to explore some of the features of the graphs.

Study carefully the graph in the following figure and the algebraic form of the **rational** function $R_1(x) = \frac{x+2}{x^2+1}$ and try to answer the following questions.



Graph of $R_1(x) = \frac{x+2}{x^2+1}$

a) For what values of x, if any, is the denominator zero ?
b) For what values of x, if any, is the denominator negative ?
c) For what values of x is the function negative ?
d) What is the value of the function when x is zero ?
e) What happens to the function if x gets very large (say 10, 100 ...) ? Substitute some values to see.

Your solution

Solution.

a) $x^2 + 1$ is never zero,

b) $x^2 + 1$ is never negative,

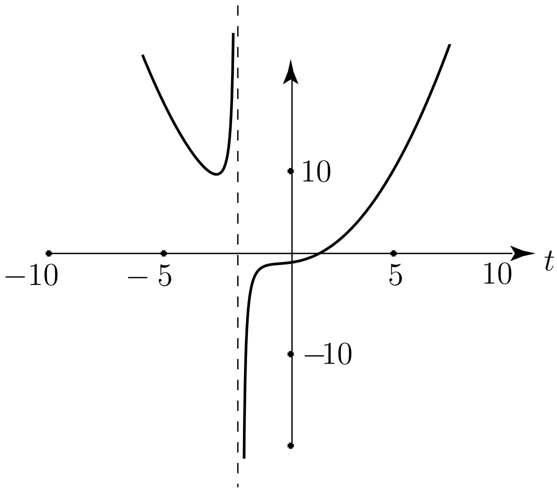
c) only when x is less than -2,

d) 2, e) R_1 approaches zero because the x^2 term in the denominator becomes very large.

Note that for large x values the graph gets closer and closer to the x axis. We say that the x axis is a **horizontal asymptote** of this graph.

Asking and answering questions such as (a) to (c) above will help you to sketch graphs of rational functions.

Study the graph and the algebraic form of the function $R_2(t) = \frac{t^3-1}{2t+3}$ carefully and try to answer the following questions. The following figure shows its graph (the solid curve).



Graph of $R_2(t) = \frac{t^3-1}{2t+3}$

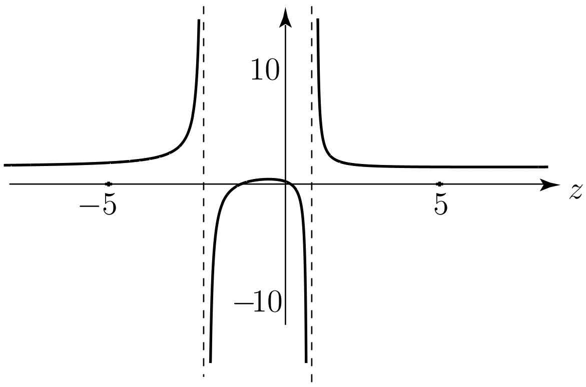
a) What is the function value when $t = 1$?
b) What is the value of the denominator when $t = -3/2$?
c) What do you think happens to the graph of the function when $t = -3/2$?

Your solution

a) 0,
b) 0
c) The function value tends to infinity, the graph becomes infinite.

Note from parts b) and c) that we must exclude the value $t = -3/2$ from the domain of this function because division by zero is not defined. At this point as you can see the graph shoots off towards very large positive values (we say it tends to positive infinity) if the point is approached from the left, and towards very large negative values (we say it tend to negative infinity) if the point is approached from the right. The dotted line in the graph of $R_2(x)$ has equation $t = -\frac{3}{2}$. It is approached by the curve as t approaches $-\frac{3}{2}$ and is known as a **vertical asymptote**.

Study the graph and the algebraic form of the function $R_3(z) = \frac{2z^2+z-1}{(z-1)(z+2)}$ carefully and try to answer the following questions. The graph of $R_3(z)$ is shown in the following figure.



Graph of $R_3(z) = \frac{2z^2+z-1}{(z-1)(z+2)}$

a) What is happening to the graph when $z = -2$ and when $z = 1$?
b) Which values should be excluded from the domain of this function ?
c) Try substituting some large values for z (e.g. 10,100 ...). What happens to R_3 as z gets large ?
d) Is there a horizontal asymptote ?
e) What is the name given to the vertical lines $z = 1$ and $z = -2$?

Your solution

a) denominator is zero, R_3 tends to infinity,
b) $z = -2$ and $z = 1$,
c) R_3 approaches the value 2,
d) $y = 2$ is a horizontal asymptote,
e) vertical asymptotes

Examples 2-4 are intended to give you some guidance so that you will be able to sketch rational functions of your own. Each function must be looked at individually but some general guidelines are given below:

- Find the value of the function when the independent variable is zero. This is generally easy to evaluate and gives you a point on the graph.

- Find values of the independent variable which make the denominator zero. These values must be excluded from the domain of the function and give rise to vertical asymptotes.

- Find values of the independent variable which make the dependent variable zero. This gives you points where the graph cuts the horizontal axis (if at all).

- Study the behaviour of the function when x is large and positive and when it is large and negative.

- Are there any vertical or horizontal asymptotes? (Oblique asymptotes may also occur but these are beyond the scope of the treatment given here).

It is particularly important for engineers to find values of the independent variable for which the denominator is zero. These values are are known as the **poles** of the rational function.

State the poles of the following rational functions:
a) $f(t) = \frac{t-3}{t+7}$
b) $F(s) = \frac{s+7}{(s+3)(s-3)}$
c) $r(x) = \frac{2x+5}{(x+1)(x+2)}$
d) $f(x) = \frac{x-1}{x^2-1}$.

Your solution

In each case we locate the poles by seeking values of the independent variable which make the denominator zero.

a) -7, b) 3 or -3, c) -1 or -2, d) $x = -1$

In each case the calculated values are the poles of the rational function. If you have access to a plotting package, plot these functions now.

Exercises

1. Explain what is meant by a rational function.
2. State the degree of the numerator and the degree of the denominator of the rational function $R(x) = \frac{3x^2+x+1}{x-1}$.
3. Explain the term 'pole' of a rational function.
4. Referring to the graphs of $R_1(x), R_2(t)$ and $R_3(z)$ (in this section), state which functions, if any, are one-to-one and which are many-to-one.
5. Without using a graphical calculator plot graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$. Comment upon whether these graphs are odd, even or neither, whether they are continuous or discontinuous, and state the position of any poles.

Answers 1. $R(x) = P(x)/Q(x)$ where P and Q are polynomials.

2. 2,1, respectively.

3. The pole is a value of the independent variable which makes the denominator zero.

4. All are many-to-one.

5. $\frac{1}{x}$ is odd, and discontinuous. Pole at $x = 0$. $\frac{1}{x^2}$ is even and discontinuous. Pole at $x = 0$.

3. The modulus function

The modulus of a number is the size of that number with no regard paid to its sign. For example the modulus of -7 is 7. The modulus of $+7$ is also 7. We can write this concisely using the modulus sign $|\,|$. So we can write $|-7| = 7$ and $|+7| = 7$. The modulus function is defined as follows:

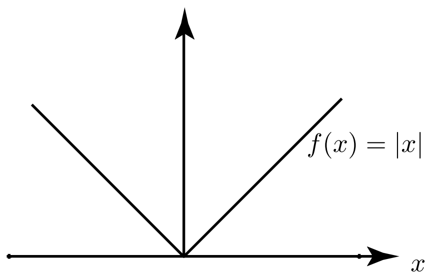
Key Point

The modulus function is defined as

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

The output from this function is simply the modulus of the input.

A graph of this function is shown in the following figure.



Graph of the modulus function $|x|$

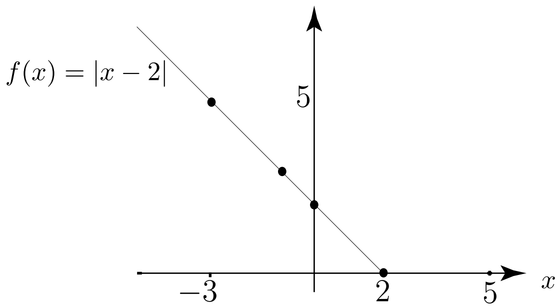
Draw up a table of values of the function $f(x) = |x - 2|$ for values of x between -3 and 5. Sketch a graph of this function.

Your solution

The table has been started. Complete it for yourself.

x	-3	-2	-1	0	1	2	3	4	5
$f(x)$	5		3	2		0			

Some points on the graph are shown in the figure. Plot your calculated points on the graph.



Exercises

1. Sketch a graph of the following functions:
a) $f(x) = 3|x|$. b) $f(x) = |x + 1|$. c) $f(x) = 7|x - 3|$.
2. Is the modulus function one-to-one or many to one ?

Answers 2. Many-to-one

4. The unit step function

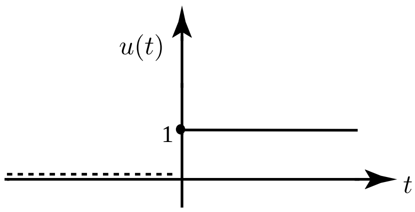
The unit step function is defined as follows:

Key Point

The unit step function $u(t)$:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

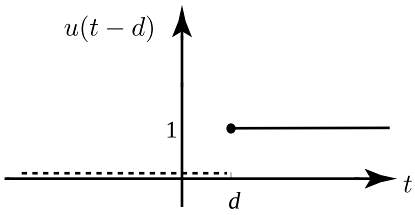
Study this definition carefully. You will see that it is defined in two parts, with one expression to be used when t is greater than or equal to 0, and another expression to be used when t is less than 0. The graph of this function is shown in the following figure. Note that the part of $u(t)$ for which $t < 0$ lies on the t-axis but, for clarity, is shown as a distinct dashed line.



Graph of the unit step function

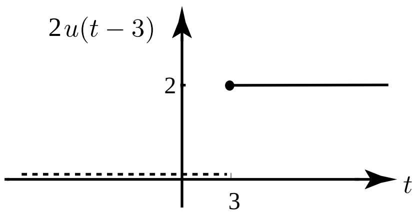
There is a jump, or discontinuity in the graph when $t = 0$. That is why we need to define the function in two parts; one part for when t is negative, and one part for when t is non-negative. The point with coordinates (0,1) is part of the function defined on $t \geq 0$.

The position of the discontinuity may be shifted to the left or right. The graph of $u(t - d)$ is shown in the next figure.



Graph of $u(t - d)$.

In the previous two figures the function takes the value 0 or 1. We can adjust the value 1 by multiplying the function by any other number we choose. The graph of $2u(t - 3)$ is shown in the next figure.

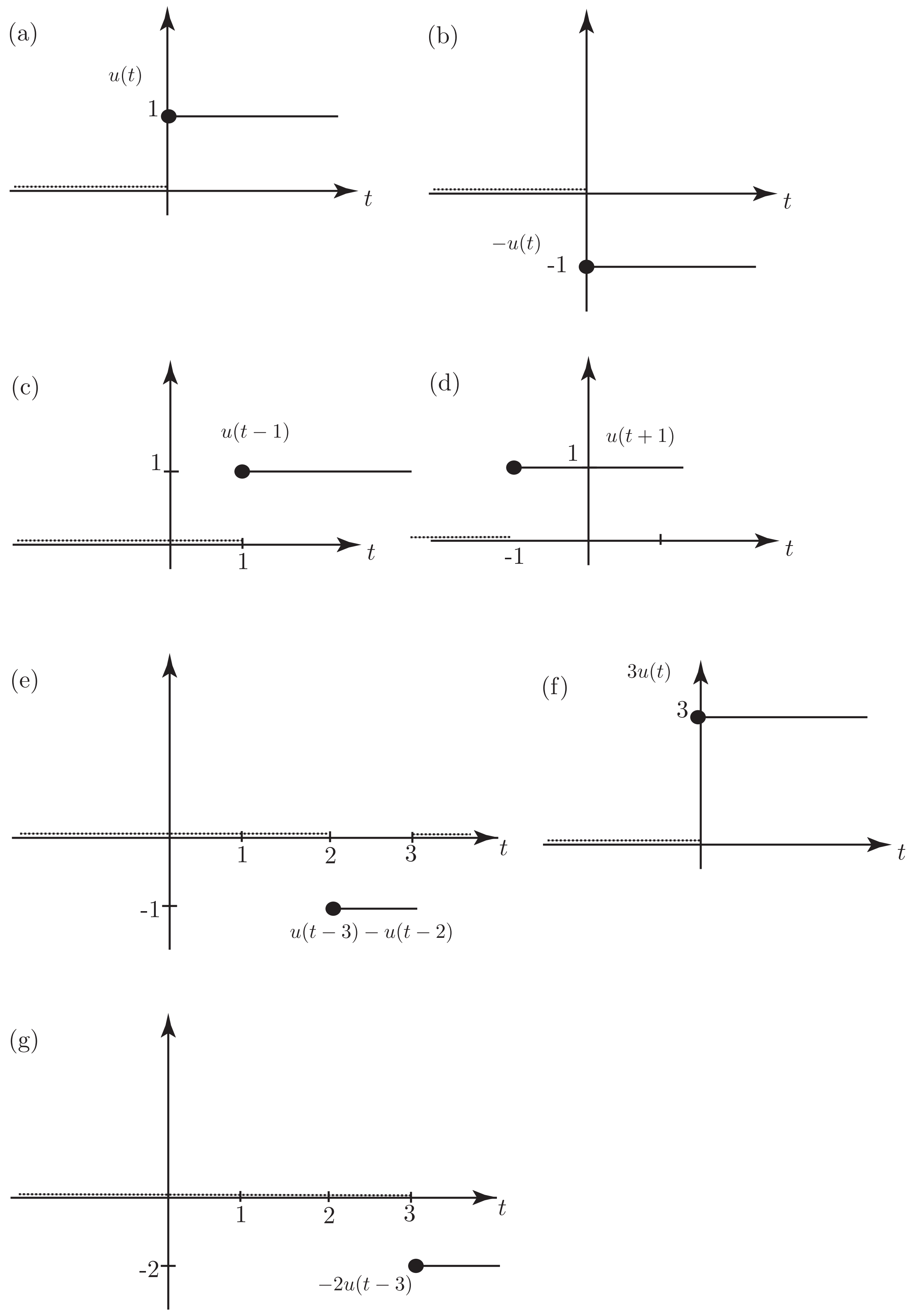


Graph of $2u(t - 3)$.

Exercises

1. Sketch graphs of the following functions:
a) $u(t)$, b) $-u(t)$, c) $u(t-1)$, d) $u(t+1)$, e) $u(t-3) - u(t-2)$, f) $3u(t)$, g) $-2u(t-3)$.

Answers 1.


(a)
u(t)
1
t
(b)
−u(t)
-1
t
(c)
u(t − 1)
1
1
t
(d)
u(t + 1)
1
-1
t
(e)
1
2
3
t
-1
u(t − 3) − u(t − 2)
(f)
3u(t)
3
t
(g)
1
2
3
t
-2
−2u(t − 3)