Contents 20

# *the laplace* **transform**

1. Causal functions
2. The transform and its inverse
3. Further Laplace transforms
4. Solving differential equations
5. The convolution theorem
6. Transfer functions

## *Learning* **outcomes**

*In this workbook you will learn what a causal function is, what the Laplace transform is, and how to obtain the transform of many commonly occurring causal functions. You will learn how the inverse Laplace transform can be obtained by using a look-up table and by using the so-called shift theorems. You will understand how to apply the Laplace transform to solve single and systems of ordinary differential equations. Finally you will gain some appreciation of transfer functions and some of their applications in solving linear systems.*

## *Time* **allocation**

*You are expected to spend approximately twelve hours of independent study on the material presented in this workbook. However, depending upon your ability to concentrate and on your previous experience with certain mathematical topics this time may vary considerably.*

# Causal Functions

## 20.1

## Introduction

The Laplace transformation is a technique employed primarily to solve constant coefficient ordinary differential equations. It is also used in modelling engineering systems. In this section we look at those functions to which the Laplace transformation is normally applied; so-called **causal or one-sided functions**. These are functions $f(t)$ of a single variable $t$ such that $f(t) = 0$ if $t < 0$. In particular we consider the simplest causal function: the unit step function (often called the Heaviside function) $u(t)$:

$$u(t) = \begin{cases} 1 & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0 \end{cases}$$

We then use this function to show how signals (functions of time $t$) may be 'switched on' and 'switched off'.

## Prerequisites

Before starting this Section you should . . .

① understand what a function is

② be able to integrate simple functions

## Learning Outcomes

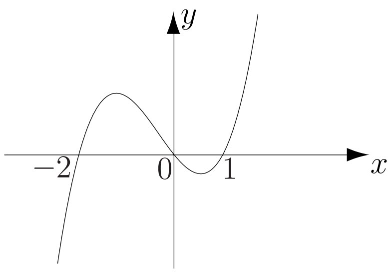
After completing this Section you should be able to . . .

✓ understand what a causal function is

✓ be able to apply the step function to 'switch on' and 'switch off' signals

## 1. Transforms and Causal Functions

Without perhaps realising it, we are used to employing transformations in mathematics. For example, we often transform problems in algebra to an equivalent problem in geometry in which our natural intuition and experience can be brought to bear. Thus, for example, if we ask: what are those values of $x$ for which $x(x-1)(x+2) > 0$ then perhaps the simplest way to solve this problem is to sketch the curve $y = x(x-1)(x+2)$ and then, by inspection, find for what values of $x$ it is positive. We obtain the following figure.




We have transformed a problem in algebra into an equivalent geometrical problem.
Clearly, by inspection of the curve, this inequality is satisfied if

$$-2 < x < 0 \quad \text{or if} \quad x > 1$$

The Laplace transform is a more complicated transformation than the simple geometric transformation considered above. What is done is to transform a function $f(t)$ of a single variable $t$ into another function $F(s)$ of a single variable $s$ through the relation:

$$F(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\mathrm{d}t.$$

The procedure is to produce, for each $f(t)$ of interest, the corresponding expression $F(s)$. As a simple example if $f(t) = \mathrm{e}^{-2t}$ then

$$\begin{aligned} F(s) &= \int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{-2t}\mathrm{d}t \\ &= \int_0^\infty \mathrm{e}^{-(s+2)t}\mathrm{d}t \\ &= \left[\frac{\mathrm{e}^{-(s+2)t}}{-(s+2)}\right]_0^\infty \\ &= 0 - \frac{\mathrm{e}^0}{-(s+2)} = \frac{1}{s+2} \end{aligned}$$

(We remind the reader that $\mathrm{e}^{-kt} \to 0$ as $t \to \infty$ if $k > 0$).

Find $F(s)$ if $f(t) = t$ using $F(s) = \displaystyle\int_0^\infty \mathrm{e}^{-st} t\,\mathrm{d}t$

**Your solution**

You should obtain $F(s) = 1/s^2$. You do this by integrating by parts:

$$F(s) = \int_0^\infty e^{-st}t\,dt = \left[t\frac{e^{-st}}{(-s)}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{(-s)}dt$$

$$= 0 + \int_0^\infty \frac{e^{-st}}{s}dt$$

$$= \left[-\frac{e^{-st}}{s^2}\right]_0^\infty = \frac{1}{s^2}$$

The integral $\displaystyle\int_0^\infty e^{-st}f(t)\mathrm{d}t$ is called the Laplace transform of $f(t)$ and is denoted by $\mathcal{L}\{f(t)\}$.

**Key Point**

Laplace Transform $\qquad \mathcal{L}\{f(t)\} = \displaystyle\int_0^\infty e^{-st}f(t)\mathrm{d}t = F(s)$

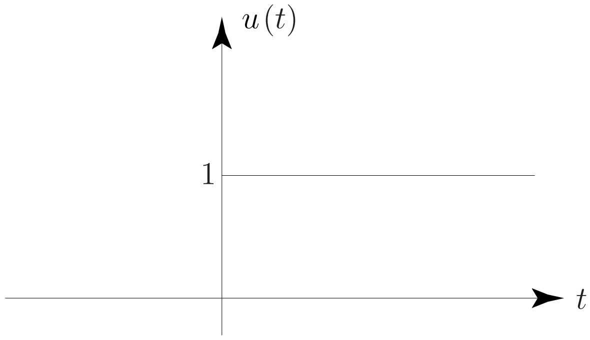
## Causal Functions

As we have seen above, the Laplace transform involves an integral with limits $t = 0$ and $\infty$. Because of this, the nature of the function being transformed, $f(t)$, when $t$ is negative is of no importance. In order to emphasize this we shall only consider so-called **causal functions** all of which take the value 0 when $t < 0$.

The simplest causal function is the Heaviside or step function denoted by $u(t)$ and defined by:

$$u(t) = \begin{cases} 1 & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0 \end{cases}$$

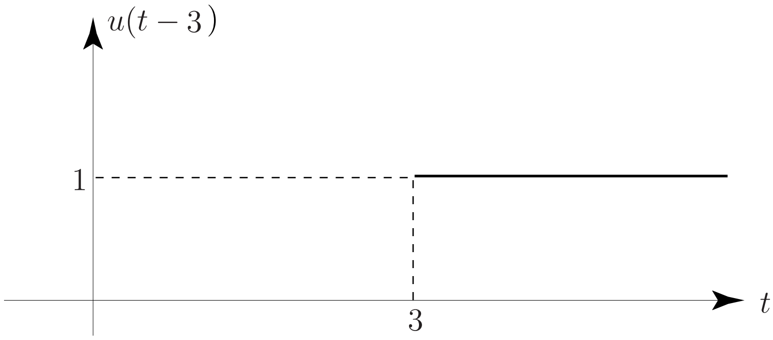
with graph



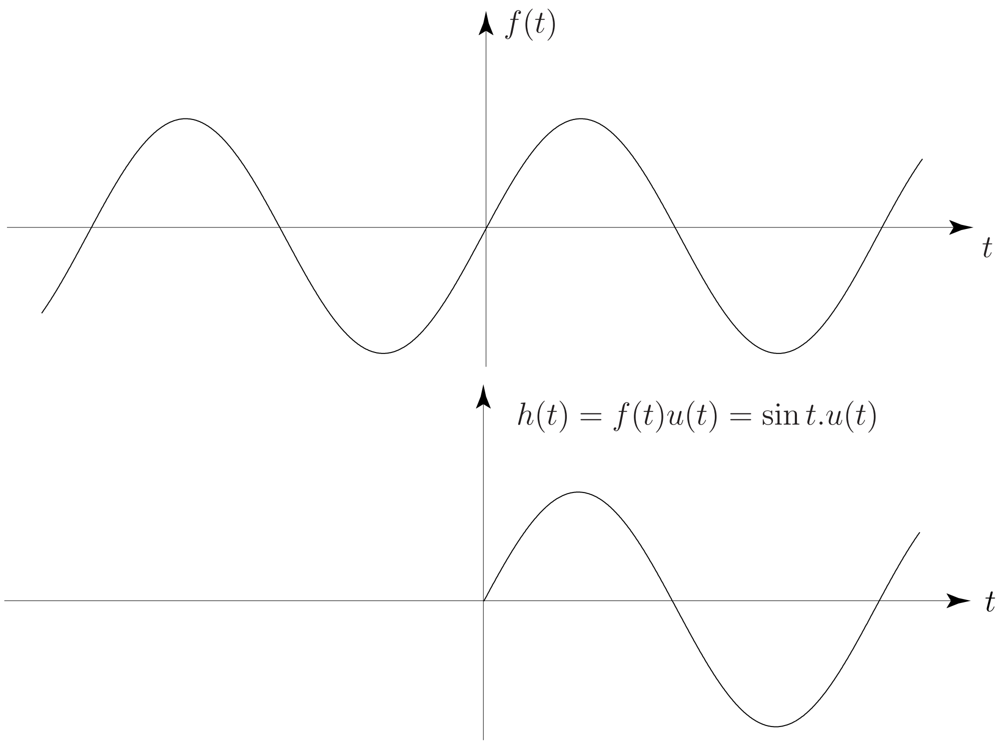

Similarly we can consider other 'step-functions'. For example, from the above definition

$$u(t-3) = \begin{cases} 1 & \text{if} \quad t-3 \geq 0 \\ 0 & \text{if} \quad t-3 < 0 \end{cases} \qquad \text{or, rearranging the inequalities :} \qquad u(t-3) = \begin{cases} 1 & \text{if} \quad t \geq 3 \\ 0 & \text{if} \quad t < 3 \end{cases}$$

with graph



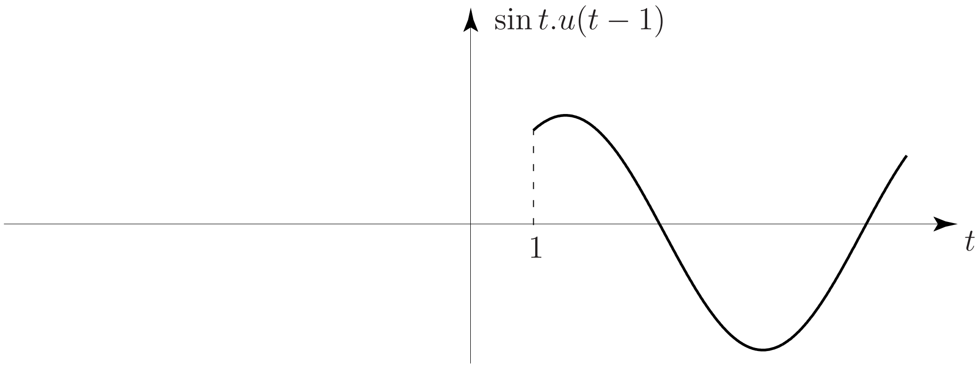

The step function has a useful property: multiplying an ordinary function $f(t)$ by the step function $u(t)$ changes it into a causal function; e.g. if $f(t) = \sin t$ then $\sin t.u(t)$ is causal.



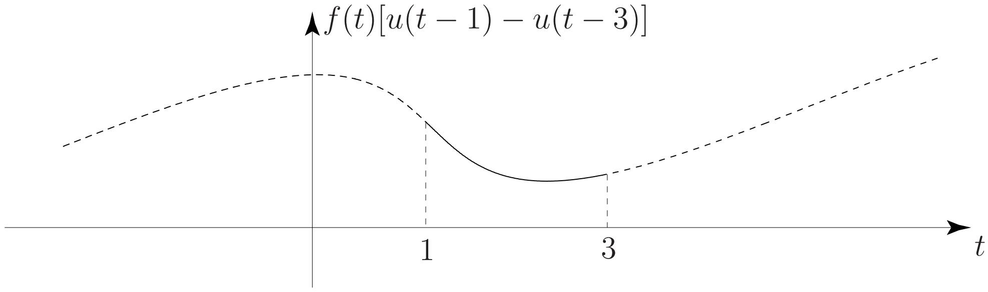
### Key Point

$f(t)u(t)$ is a causal function

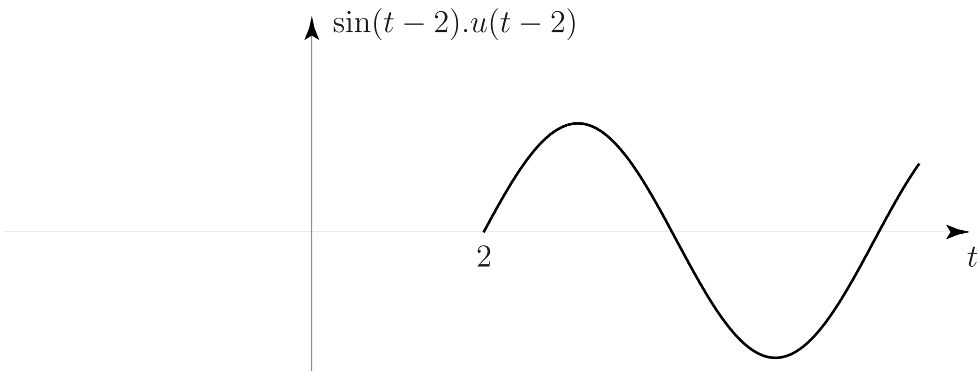
The step function can be used to 'switch on' functions at other values of $t$ (which we will normally interpret as the time). For example $u(t-1)$ has the value 1 if $t \geq 1$ and 0 otherwise so that $\sin t.u(t-1)$ is described by the following (solid) curve:




The step function can also be used to 'switch-off' signals. For example $f(t)\,[u(t-1)-u(t-3)]$ which is described by the solid curve below, switches on at $t=1$ (for then $u(t-1)-u(t-3)$ takes the value 1). The signal remains 'on' as $1 \leq t \leq 3$ and then switches 'off' when $t > 3$ (for then $u(t-1)-u(t-3) = 1-1 = 0$)



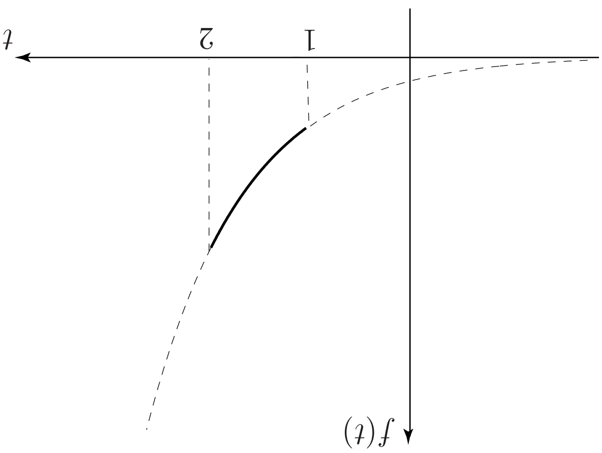

However, if we have an expression $f(t-a)u(t-a)$ then this is the function $f(t)$ translated along the $t$-axis through a time $a$. For example $\sin(t-2).u(t-2)$ is simply the causal sine curve $\sin t.u(t)$ shifted to the right by two units as described in the following diagram.



Sketch the curve $f(t) = \mathrm{e}^t(u(t-1) - u(t-2))$.

**Your solution**

You should obtain




This is obtained since, if $t < 1$ then $t - 1 < 0$ and $t - 2 < 0$ and so

$$u(t-1) = 0, \qquad u(t-2) = 0 \qquad \text{leading to} \quad f(t) = 0$$

Also if $1 < t < 2$ then $t - 1 > 0$ and $t - 2 < 0$ so

$$u(t-1) = 1 \qquad \text{and} \qquad u(t-2) = 0 \qquad \text{implying} \quad f(t) = \mathrm{e}^t \quad \text{for this range of } t\text{−values.}$$

Finally if $t > 2$ then $t - 1 > 0$ and $t - 2 > 0$ and so

$$u(t-1) = 1, \qquad u(t-2) = 1 \qquad \text{giving} \quad f(t) = \mathrm{e}^t(1-1) = 0.$$

## 2. Properties of Causal Functions

Even though a function $f(t)$ may be causal we shall still often use the step function $u(t)$ to emphasize its causality and write $f(t)\ u(t)$. The following properties are easily verified.

(i) The sum of casusal functions is causal:

$$f(t)u(t) + g(t)u(t) = [f(t) + g(t)]\, u(t)$$

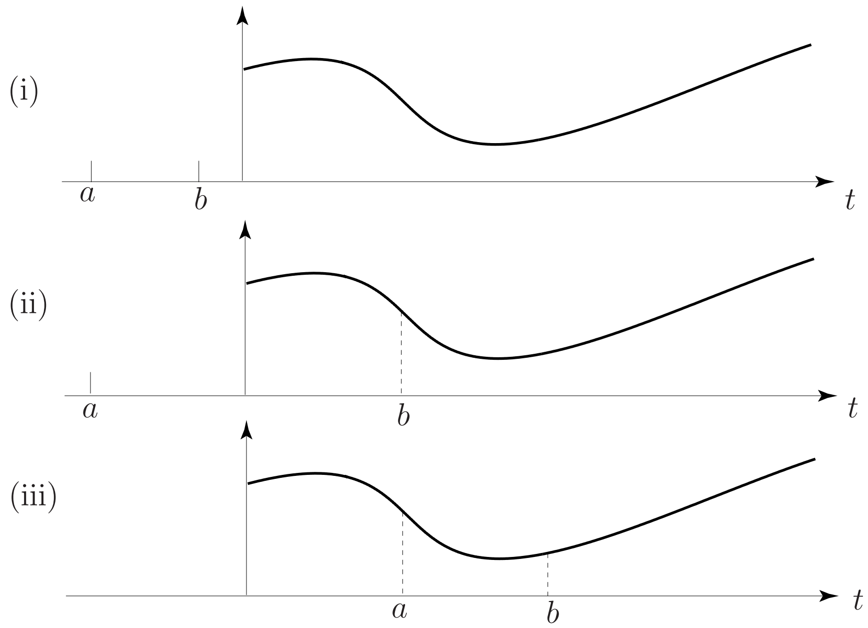
(ii) The product of causal functions is causal:

$$\{f(t)u(t)\}\ \{g(t)u(t)\} = f(t)g(t).u(t)$$

(iii) The derivative of a causal function is causal:

$$\frac{\mathrm{d}}{\mathrm{d}t}\{f(t)u(t)\} = \frac{\mathrm{d}f}{\mathrm{d}t}.u(t)$$

The definite integral of a causal function needs care. Consider $\displaystyle\int_a^b f(t)u(t)\mathrm{d}t$ where $a < b$. There are 3 cases to consider (i) $b < 0$ (ii) $a < 0,\ \ b > 0$ and (iii) $a > 0$ which are described in the following diagram:




(i) If $b < 0$ then $t < 0$ and so $u(t) = 0$

$$\therefore \qquad \int_a^b f(t)u(t)\mathrm{d}t = 0$$

(ii) If $a < 0,\ b > 0$ then

$$\begin{aligned}\int_a^b f(t)u(t)\mathrm{d}t &= \int_a^0 f(t)u(t)\mathrm{d}t + \int_0^b f(t)u(t)\mathrm{d}t \\ &= 0 + \int_0^b f(t)u(t)\mathrm{d}t \\ &= \int_0^b f(t)\mathrm{d}t\end{aligned}$$

since, in the first integral $t < 0$ and so $u(t) = 0$ whereas, in the second integral $t > 0$ and so $u(t) = 1$.

(iii) If $a > 0$ then

$$\int_a^b f(t)u(t)\mathrm{d}t = \int_a^b f(t)\mathrm{d}t$$

since $t > 0$ and so $u(t) = 1$.

If $f(t) = (\mathrm{e}^{-t} + t)u(t)$ then find $\dfrac{df}{\mathrm{d}t}$ and $\displaystyle\int_{-3}^{4} f(t)\mathrm{d}t$

**Your solution**

Find the derivative first

$$\frac{df}{\mathrm{d}t} = (-\mathrm{e}^{-t} + 1)u(t)$$

**Your solution**

Now obtain another integral representing $\displaystyle\int_{-3}^{4} f(t)\mathrm{d}t$

You should obtain $\displaystyle\int_0^4 (\mathrm{e}^{-t} + t)\mathrm{d}t$ since

$$\int_{-3}^{4} f(t)\mathrm{d}t = \int_{-3}^{4} (\mathrm{e}^{-t} + t)u(t)\mathrm{d}t$$
$$= \int_{0}^{4} (\mathrm{e}^{-t} + t)\mathrm{d}t$$

since in the range $t = -3$ to $t = 0$ the step function $u(t) = 0$ and so that part of the integral is zero. In the other part of the integral $u(t) = 1$.

**Your solution**

Now complete the integration

You should obtain 8.9817 (to 4d.p.) since

$$\int_{-4}^{0}(e^{-t}+t)\,dt \;=\; \left[-e^{-t}+\frac{t^2}{2}\right]_{-4}^{0} = (-e^{-4}+8)-(-1)$$

$$= \;-e^{-4}+9 \approx 8.9817$$

## Exercises

1. Find the derivative of $(t^3+\sin t)\ u(t)$.

2. Find the area under the curve $(t^3+\sin t)u(t)$ between $t=-3$ and $t=1$.

3. Find the area under the curve $\dfrac{1}{(t+3)}\ [u(t-1)-u(t-3)]$ between $t=-2$ and $t=2.5$

**Answers** 1. $(3t^2+\cos t)u(t)$.

2. 0.7097

3. 0.3185