The Transform and its Inverse

20.2

Introduction

In this section we formally introduce the Laplace transform. The transform is only applied to causal functions which were introduced in section 20.1. We find the Laplace transform of many commonly occurring 'signals'and produce a table of standardLaplace transforms.

We also consider the inverse Laplace transform. To begin with, the inverse Laplace transform is obtained 'by inspection' using a table of transforms. This approach is developed by employing techniques such as partial fractions and completing the square.

Prerequisites

Before starting this Section you should ...

① understand what a causal function is

② understand how to use partial fractions

③ be familiar with integration by parts

④ understand the technique of completing the square

Learning Outcomes

After completing this Section you should be able to ...

✓ find the Laplace transform of many commonly occurrring causal functions

✓ obtain the inverse Laplace transform using
(i) a table of transforms,
(ii) partial fractions,
(iii) the method of completing the square

1. The Laplace Transformation

If $f(t)$ is a **causal function** then the Laplace transform of $f(t)$ is written $\mathcal{L}\{f(t)\}$ and defined by:

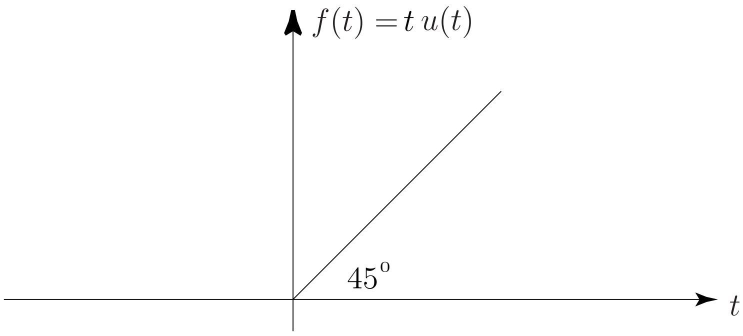
$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt.$$

Clearly, once the integral is performed and the limits substituted the resulting expression will involve the s-parameter alone since the dependence upon t is removed in the integration process. This resulting expression in s is denoted by $F(s)$; its precise form is dependent upon the form taken by $f(t)$

Key Point

The Laplace Transform $\qquad \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt \equiv F(s)$

To begin, we determine the Laplace transform of some simple causal functions. For example, if we consider the **ramp function** $f(t) = t.u(t)$ with graph



we find:

$$\begin{aligned}
\mathcal{L}\{t\ u(t)\} &= \int_0^{\infty} e^{-st} t\ u(t)dt \\
&= \int_0^{\infty} e^{-st} t\ dt \qquad \text{since in the range of the integral } u(t) = 1 \\
&= \left[\frac{t\ e^{-st}}{(-s)}\right]_0^{\infty} - \int_0^{\infty} \frac{e^{-st}}{(-s)}\ dt \qquad \text{using integration by parts} \\
&= \left[\frac{te^{-st}}{(-s)}\right]_0^{\infty} - \left[\frac{e^{-st}}{(-s)^2}\right]_0^{\infty}
\end{aligned}$$

Now we have the problem of substituting in the limits of integration. The only problem arises with the upper limit (∞). We shall always assume that the parameter s is so chosen that no contribution ever arises from the upper limit (∞). In this particular case we need only demand that s is real and positive Using this 'rule of thumb':

$$\begin{aligned}\mathcal{L}\{t\ u(t)\} &= [0-0] - \left[0 - \left(\frac{1}{(-s)^2}\right)\right] \\ &= \frac{1}{s^2}\end{aligned}$$

Thus, if $f(t) = t\ u(t)$ then $F(s) = 1/s^2$.
A similar, but more tedius calculation, yields the result that if $f(t) = t^n u(t)$ in which n is a positive integer:

$$\mathcal{L}\{t^n u(t)\} = \frac{n!}{s^{n+1}}$$

We remember $\quad n! \equiv n(n-1)(n-2)\ldots(3)(2)(1)$.

Find the Laplace transform of the step function $u(t)$.

Your solution

Begin by obtaining the Laplace integral

You should obtain $\int_0^\infty e^{-st}dt$ since

$$\mathcal{L}\{u(t)\} = \int_0^\infty e^{-st}u(t)\ dt = \int_0^\infty e^{-st}dt$$

and, in the range of integration, $t > 0$ and so $u(t) = 1$.

Your solution

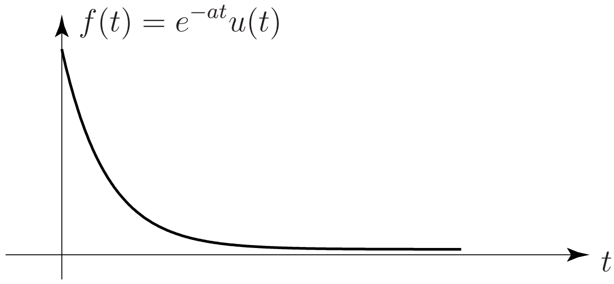
Now complete the integration

You should have obtained:

$$\begin{aligned}\mathcal{L}\{u(t)\} &= \int_0^\infty e^{-st}dt \\ &= \left[\frac{e^{-st}}{(-s)}\right]_0^\infty = 0 - \left[\frac{1}{(-s)}\right] = \frac{1}{s}\end{aligned}$$

where, again, we have assumed the contribution from the upper limit is zero.

As a second example consider the decaying exponential $f(t) = \mathrm{e}^{-at}u(t)$ where a is a positive constant. This function has graph:



In this case,

$$\begin{aligned}
\mathcal{L}\{\mathrm{e}^{-at}u(t)\} &= \int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{-at}\mathrm{d}t \\
&= \int_0^\infty \mathrm{e}^{-(s+a)t}\mathrm{d}t \\
&= \left[\frac{\mathrm{e}^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{1}{s+a} \qquad \text{zero contribution from the upper limit}
\end{aligned}$$

Therefore, if $f(t) = \mathrm{e}^{-at}u(t)$ then $F(s) = \dfrac{1}{s+a}$.

Following this approach we can develop a table of Laplace transforms which records, for each causal function $f(t)$, its corresponding transform function $F(s)$. Table 1 gives a limited table of transforms.

The Linearity Property of the Laplace Transformation

If $f(t)$ and $g(t)$ are causal functions and c_1, c_2 are constants then

$$\begin{aligned}
\mathcal{L}\{c_1f(t) + c_2g(t)\} &= \int_0^\infty \mathrm{e}^{-st}[c_1f(t) + c_2g(t)]\mathrm{d}t \\
&= c_1\int_0^\infty \mathrm{e}^{-st}f(t)\mathrm{d}t + c_2\int_0^\infty \mathrm{e}^{-st}g(t)\mathrm{d}t \\
&= c_1\mathcal{L}\{f(t)\} + c_2\mathcal{L}\{g(t)\}
\end{aligned}$$

Key Point

$$\mathcal{L}\{c_1f(t) + c_2g(t)\} = c_1\mathcal{L}\{f(t)\} + c_2\mathcal{L}\{g(t)\}$$

That is, the Laplace transform of a linear sum of causal functions is a linear sum of Laplace transforms. For example,

$$\begin{aligned}
\mathcal{L}\{2\cos t\,.\,u(t) - 3t^2u(t)\} &= 2\mathcal{L}\{\cos t\,.\,u(t)\} - 3\mathcal{L}\{t^2u(t)\} \\
&= 2\left(\frac{s}{s^2+1}\right) - 3\left(\frac{2}{s^3}\right)
\end{aligned}$$

Table 1. Table of transforms

Causal function	Laplace transform
$f(t)$	$F(s)$
$u(t)$	$\dfrac{1}{s}$
$t^n u(t)$	$\dfrac{n!}{s^{n+1}}$
$e^{-at}u(t)$	$\dfrac{1}{s+a}$
$\sin at\,.\,u(t)$	$\dfrac{a}{s^2+a^2}$
$\cos at\,.\,u(t)$	$\dfrac{s}{s^2+a^2}$
$e^{-at}\sin bt\,.\,u(t)$	$\dfrac{b}{(s+a)^2+b^2}$
$e^{-at}\cos bt\, u(t)$	$\dfrac{s+a}{(s+a)^2+b^2}$

Obtain the Laplace transforms of the hyperbolic functions (i) $\sinh at$, (ii) $\cosh at$.

Your solution

(i) Begin by expressing $\sinh at$ in terms of exponential functions.

$\sinh at = \frac{1}{2}(e^{at} - e^{-at})$

Your solution

Now use the linearity property to obtain the Laplace transform of the causal function $\sinh at.u(t)$

You should obtain $a/(s^2 - a^2)$ since

$$\mathcal{L}\{\sinh at.u(t)\} = \mathcal{L}\left\{\frac{e^{at} - e^{-at}}{2}.u(t)\right\} = \frac{1}{2}\mathcal{L}\{e^{at}.u(t)\} - \frac{1}{2}\mathcal{L}\{e^{-at}.u(t)\}$$

$$= \frac{1}{2}\left[\frac{1}{s-a}\right] - \frac{1}{2}\left[\frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{2a}{(s-a)(s+a)}\right] = \frac{a}{s^2 - a^2}$$

Your solution

(ii)

You should obtain $\dfrac{s}{s^2 - a^2}$ since

$$\mathcal{L}\{\cosh at.u(t)\} = \mathcal{L}\left\{\frac{e^{at} + e^{-at}}{2}.u(t)\right\} = \frac{1}{2}\mathcal{L}\{e^{at}.u(t)\} + \frac{1}{2}\mathcal{L}\{e^{-at}.u(t)\}$$

$$= \frac{1}{2}\left[\frac{1}{s-a}\right] + \frac{1}{2}\left[\frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{(s-a)(s+a)}\right] = \frac{s}{s^2 - a^2}$$

Find the Laplace transform of the **delayed step-function** $u(t-a)$, $a > 0$

Your solution

Write it here in terms of an integral

You should obtain $\mathcal{L}\{u(t-a)\} = \int_a^\infty e^{-st}dt$ (note the change in the lower limit) since:

$$\begin{aligned}\mathcal{L}\{u(t-a)\} &= \int_0^\infty e^{-st}u(t-a)dt \\ &= \int_0^a e^{-st}u(t-a)dt + \int_a^\infty e^{-st}u(t-a)dt\end{aligned}$$

In the first integral $0 < t < a$ and so $(t-a) < 0$, therefore $u(t-a) = 0$. In the second integral $a < t < \infty$ and so $(t-a) > 0$, therefore $u(t-a) = 1$. Hence

$$\mathcal{L}\{u(t-a)\} = 0 + \int_a^\infty e^{-st}dt.$$

Your solution

Now complete the integration

$$\begin{aligned}\mathcal{L}\{u(t-a)\} &= \int_a^\infty e^{-st}dt \\ &= \left[\frac{e^{-st}}{(-s)}\right]_a^\infty = \frac{e^{-sa}}{s}\end{aligned}$$

Exercises

1. Determine the Laplace transform of the following functions.
(i) $e^{-3t}u(t)$ (ii) $u(t-3)$ (iii) $e^{-t}\sin 3t.u(t)$ (iv) $(5\cos 3t - 6t^3).u(t)$

Answers 1. (i) $\dfrac{1}{s+3}$ (ii) $\dfrac{e^{-3s}}{s}$ (iii) $\dfrac{3}{(s+1)^2+9}$ (iv) $\dfrac{5s}{s^2+9} - \dfrac{36}{s^4}$

2. The Inverse Laplace Transformation

The Laplace transform takes a causal function $f(t)$ and transforms it into a function of s, $F(s)$:

$$\mathcal{L}\{f(t)\} \equiv F(s)$$

The inverse Laplace transform operator is denoted by $\mathcal{L}^{-1}$ and involves recovering the original causal function $f(t)$. That is,

Key Point

Inverse Laplace transform $\quad \mathcal{L}^{-1}\{F(s)\} = f(t) \quad$ where $\quad \mathcal{L}\{f(t)\} = F(s)$

For example, using standard transforms from Table 1

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t\,.\,u(t) \text{ since } \mathcal{L}\{\cos 2t\,.\,u(t)\} = \frac{s}{s^2+4}.$$

Also

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2}\right\} = 3t\,u(t) \text{ since } \mathcal{L}\{3t\,u(t)\} = \frac{3}{s^2}.$$

Because the Laplace transform is linear it follows that the inverse Laplace transform is also linear, so if $c_1,\ c_2$ are constants:

Key Point

$$\mathcal{L}^{-1}\{c_1F(s) + c_2G(s)\} = c_1\mathcal{L}^{-1}\{F(s)\} + c_2\mathcal{L}^{-1}\{G(s)\}$$

For example, to find the inverse Laplace transform of $\dfrac{2}{s^4} - \dfrac{6}{s^2+4}$ we have

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{2}{s^4} - \frac{6}{s^2+4}\right\} &= \tfrac{2}{6}\mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} - 3\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \\ &= \frac{1}{3}t^3u(t) - 3\sin 2t\,.\,u(t) \qquad \text{from the table of transforms}\end{aligned}$$

Note that the fractions have had to be manipulated slightly in order that the expression accord precisely with the expressions in Table 1. Although the inverse Laplace transform can be examined at a deeper mathematical level we shall be content with this simple minded approach to finding inverse Laplace transforms by using the table of Laplace transforms. However, even this approach is not always straightforward and algebraic manipulation is often required before an inverse Laplace transform can be found. Here we consider two standard rearrangements which often occur.

Inverting through the use of partial fractions

The function

$$F(s) = \frac{1}{(s-1)(s+2)}$$

does not appear in our table of transforms and so we cannot, by inspection, write down the inverse Laplace transform. However, by using partial fractions we see that

$$F(s) = \frac{1}{(s-1)(s+2)} = \frac{\frac{1}{3}}{s-1} - \frac{\frac{1}{3}}{s+2}$$

and so

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+2)}\right\} &= \mathcal{L}^{-1}\left\{\frac{\frac{1}{3}}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{\frac{1}{3}}{s+2}\right\} \\ &= \tfrac{1}{3}\mathrm{e}^{t} - \tfrac{1}{3}\mathrm{e}^{-2t}\end{aligned}$$

Find the inverse Laplace transform of $\dfrac{3}{(s-1)(s^2+1)}$.

Your solution

Begin by using partial fractions to write the given expression in a more suitable form

$$\frac{3}{(s-1)(s^2+1)} = \frac{\frac{3}{2}}{s-1} - \frac{\frac{3}{2}s + \frac{3}{2}}{s^2+1}$$

Your solution

Now continue to obtain the inverse

You should obtain:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{3}{(s-1)(s^2+1)}\right\} &= \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= \frac{3}{2}\left[\mathrm{e}^{t} - \cos t - \sin t\right] u(t)\end{aligned}$$

Inverting using completion of the square

The function:

$$F(s) = \frac{4s}{s^2 + 2s + 5}$$

does not appear in the table of transforms and, again, needs amending before we can find its inverse transform. In this case, because $s^2 + 2s + 5$ does not have nice factors, we complete the square in the denominator:

$$s^2 + 2s + 5 \equiv (s+1)^2 + 4$$

and so

$$F(s) = \frac{4s}{s^2 + 2s + 5} = \frac{4s}{(s+1)^2 + 4}$$

Now the numerator needs amending slightly to enable us to use the table of standard transforms:

$$\begin{aligned} F(s) = \frac{4s}{(s+1)^2+4} \quad &= \quad 4\left\{\frac{s+1-1}{(s+1)^2+4}\right\} \\ &= \quad 4\left\{\frac{s+1}{(s+1)^2+4} - \frac{1}{(s+1)^2+4}\right\} \\ &= \quad \frac{4(s+1)}{(s+1)^2+4} - 2\left[\frac{2}{(s+1)^2+4}\right] \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{L}^{-1}\{F(s)\} \quad &= \quad 4\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} - 2\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\} \\ &= \quad 4\mathrm{e}^{-t}\cos 2t\,.\,u(t) - 2\mathrm{e}^{-t}\sin 2t\,.\,u(t) \\ &= \quad \mathrm{e}^{-t}[4\cos 2t - 2\sin 2t]u(t) \end{aligned}$$

Find the inverse Laplace transform of $\dfrac{3}{s^2 - 4s + 6}$.

Your solution

Begin by completing the square in the denominator of this expression

$$\frac{3}{s^2 - 4s + 6} = \frac{3}{(s-2)^2 + 2}$$

Your solution

Now continue to obtain the inverse

You should obtain:

$$\mathcal{L}^{-1}\left\{\frac{3}{(s-2)^2+2}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{\sqrt{2}}\left[\frac{\sqrt{2}}{(s-2)^2+2}\right]\right\}$$

$$= \frac{3}{\sqrt{2}}e^{2t}\sin\sqrt{2}t.u(t)$$

Exercises

Determine the inverse Laplace transforms of the following functions.

(i) $\dfrac{10}{s^4}$ (ii) $\dfrac{s-1}{s^2+8s+17}$ (iii) $\dfrac{3s-7}{s^2+9}$ (iv) $\dfrac{3s+3}{(s-1)(s+2)}$ (v) $\dfrac{s+3}{s^2+4s}$

(vi) $\dfrac{2}{(s+1)(s^2+1)}$

Answers (i) $\frac{10}{6}t^3$ (ii) $e^{-4t}\cos t - 5e^{-4t}\sin t$ (iii) $3\cos 3t - \frac{7}{3}\sin 3t$ (iv) $2e^t + e^{-2t}$

(v) $\frac{3}{4}u(t) + \frac{1}{4}e^{-4t}u(t)$ (vi) $(e^{-t} - \cos t + \sin t)u(t)$