Further Laplace Transforms

20.3

Introduction

In this Section we introduce the first and second shift theorems which will ease the determination of Laplace and inverse Laplace transforms of more complicated causal functions.

Then we obtain the Laplace transform of derivatives of causal functions. This will allow us, in the next Section, to apply the Laplace transform in the solution of ordinary differential equations.

Finally, we introduce the delta function and obtain its Laplace transform. The delta function is often needed to model the effect on a system of a forcing function which acts for a very short time.

Prerequisites

Before starting this Section you should . . .

① be able to find Laplace and inverse Laplace transforms of simple causal functions

② be familiar with integration by parts

③ understand what an initial-value problem is

Learning Outcomes

After completing this Section you should be able to . . .

✓ use the shift theorems to obtain Laplace and inverse Laplace transforms

✓ take the Laplace transform of the derivative of a causal function

1. The First and Second Shift Theorems

The shift theorems enable an even wider range of Laplace transforms to be easily obtained from the transforms we have already found and also enable a significantly wider range of inverse transforms to be found.

The first shift theorem

If $f(t)$ is a causal function with Laplace transform $F(s)$, i.e. $\mathcal{L}\{f(t)\} = F(s)$, then as we shall see, the Laplace transform of $\mathrm{e}^{-at}f(t)$, where a is a given constant, can easily be found in terms of $F(s)$.
Using the definition of the Laplace transform:

$$\begin{aligned}\mathcal{L}\{\mathrm{e}^{-at}f(t)\} &= \int_0^\infty \mathrm{e}^{-st}\left[\mathrm{e}^{-at}f(t)\right]\mathrm{d}t \\ &= \int_0^\infty \mathrm{e}^{-(s+a)t}f(t)\mathrm{d}t\end{aligned}$$

But if

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty \mathrm{e}^{-st}f(t)\mathrm{d}t$$

then by simply replacing 's' by '$s+a$' on both sides:

$$F(s+a) = \int_0^\infty \mathrm{e}^{-(s+a)t}f(t)\mathrm{d}t$$

That is, the parameter s is shifted to the value $s+a$. We have then the statement of the first shift theorem:

Key Point

If $\quad \mathcal{L}\{f(t)\} = F(s) \quad$ then $\quad \mathcal{L}\{\mathrm{e}^{-at}f(t)\} = F(s+a)$

For example, we already know (from tables) that

$$\mathcal{L}\{t^3u(t)\} = \frac{6}{s^4}$$

and so, by the first shift theorem:

$$\mathcal{L}\{\mathrm{e}^{-2t}t^3u(t)\} = \frac{6}{(s+2)^4}$$

Use the first shift theorem to determine $\mathcal{L}\{e^{2t}\cos 3t.u(t)\}$

Your solution

You should obtain $\dfrac{s-2}{(s-2)^2+9}$ since $\mathcal{L}\{\cos 3t.u(t)\} = \dfrac{s}{s^2+9}$ and so by the first shift theorem (with $a=-2$)

$$\mathcal{L}\{e^{2t}\cos 3t.u(t)\} = \frac{s-2}{(s-2)^2+9}$$

obtained by simply replacing 's' by '$s-2$'.

We can also employ the first shift theorem to determine some inverse Laplace transforms.

Find the inverse Laplace transform of $F(s) = \dfrac{3}{s^2-2s-8}$

Your solution

Begin by completing the square in the denominator

$$\frac{3}{s^2-2s-8} = \frac{3}{(s-1)^2-9}$$

Your solution

Realising that $\mathcal{L}\{\sinh 3t\ u(t)\} = \dfrac{3}{s^2-9}$, complete the inversion using the first shift theorem

You should obtain

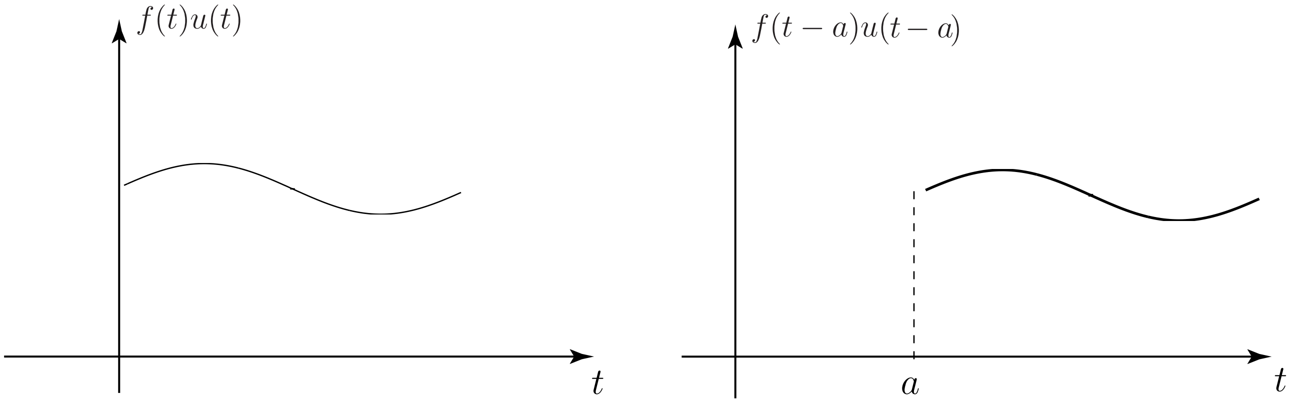
$$\mathcal{L}^{-1}\left\{\frac{3}{(s-1)^2-9}\right\} = e^t \sinh 3t\ u(t)$$

Here, in the notation of the shift theorem:

$$f(t) = \sinh 3t\ u(t) \qquad F(s) = \frac{3}{s^2-9} \qquad \text{and} \qquad a=-1$$

The second shift theorem

The second shift theorem is similar to the first except that, in this case, it is the time-variable that is shifted not the s-variable. Consider a causal function $f(t)u(t)$ which is shifted to the right by amount a, that is, the function $f(t-a)u(t-a)$ where $a > 0$. The following figure illustrates the two causal functions.



The Laplace transform of the shifted function is easily obtained:

$$\begin{aligned}\mathcal{L}\{f(t-a)u(t-a)\} &= \int_0^\infty \mathrm{e}^{-st} f(t-a)u(t-a)\mathrm{d}t \\ &= \int_a^\infty \mathrm{e}^{-st} f(t-a)\mathrm{d}t\end{aligned}$$

(Note the change in the lower limit from 0 to a resulting from the step function switching on at $t = a$). We can re-organise this integral by making the substitution $x = t - a$. Then

$$\mathrm{d}t = \mathrm{d}x \quad \text{and when } t = a, x = 0 \quad \text{and when } t = \infty \text{ then } x = \infty.$$

Therefore

$$\begin{aligned}\int_a^\infty \mathrm{e}^{-st} f(t-a)\mathrm{d}t &= \int_0^\infty \mathrm{e}^{-s(x+a)} f(x)\mathrm{d}x \\ &= \mathrm{e}^{-sa} \int_0^\infty \mathrm{e}^{-sx} f(x)\mathrm{d}x\end{aligned}$$

The final integral is simply the Laplace transform of $f(x)$, which we know is $F(s)$ and so, finally, we have the statement of the second shift theorem:

Key Point

If $\quad \mathcal{L}\{f(t)\} = F(s) \quad$ then $\quad \mathcal{L}\{f(t-a)u(t-a)\} = \mathrm{e}^{-sa}F(s)$

Obviously, this theorem has its uses in finding the Laplace transform of time-shifted causal functions but it is also of considerable use in finding inverse Laplace transforms since, using the inverse formulation of the theorem above:

Key Point

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ then $\mathcal{L}^{-1}\{e^{-sa}F(s)\} = f(t-a)u(t-a)$

Find the inverse Laplace transforms of (i) $\dfrac{e^{-3s}}{s^2}$ (ii) $\dfrac{s}{s^2-2s+2}$

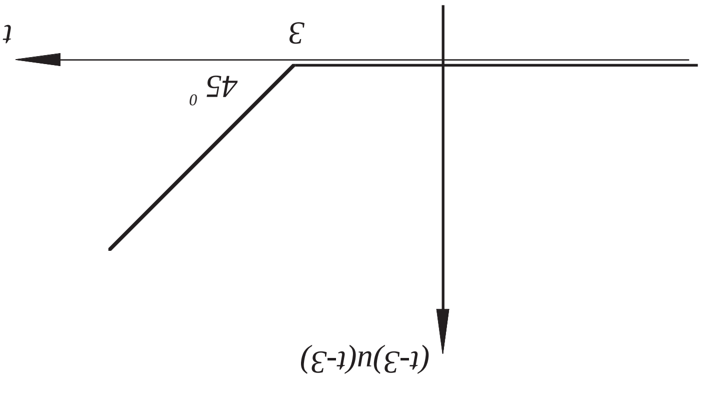
Your solution

(i)

(i) You should obtain $(t-3)u(t-3)$ for the following reasons. We know that the inverse Laplace transform of $1/s^2$ is $tu(t)$ and so, using the second shift theorem (with $a=3$), we have

$$\mathcal{L}^{-1}\left\{e^{-3s}\frac{1}{s^2}\right\} = (t-3)u(t-3)$$

This function is graphed in the following figure:



Your solution

(ii)

You should obtain $e^t(\cos t + \sin t)$. To obtain this, complete the square in the denominator: $s^2 - 2s + 2 = (s-1)^2 + 1$ and so

$$\frac{s}{s^2-2s+2} = \frac{s}{(s-1)^2+1} = \frac{(s-1)+1}{(s-1)^2+1} = \frac{s-1}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}$$

Now, using the first shift theorem

$$\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\} = e^t \cos t.u(t) \qquad \text{since} \qquad \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t.u(t)$$

and

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\} = e^t \sin t.u(t) \qquad \text{since} \qquad \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t.u(t)$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2-2s+2}\right\} = e^t(\cos t + \sin t)u(t)$$

2. The Laplace transform of a derivative

Here we consider not a causal function $f(t)$ directly but its derivatives $\frac{df}{dt}, \frac{d^2f}{dt^2}, \ldots$ (which are also causal). The Laplace transform of derivatives will be invaluable when we apply the Laplace transform to the solution of constant coefficient ordinary differential equations.

If $\mathcal{L}\{f(t)\}$ is $F(s)$ then we shall seek an expression for $\mathcal{L}\{\frac{df}{dt}\}$ in terms of the function $F(s)$.

Now, by the definition of the Laplace transform

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = \int_0^\infty e^{-st}\frac{df}{dt}dt$$

This integral can be simplified using integration by parts:

$$\begin{aligned}\int_0^\infty e^{-st}\frac{df}{dt}dt &= \left[e^{-st}f(t)\right]_0^\infty - \int_0^\infty (-s)e^{-st}f(t)dt \\ &= -f(0) + s\int_0^\infty e^{-st}f(t)dt\end{aligned}$$

(As usual,we assume that contributions arising from the upper limit ∞ are zero). The integral that remains is precisely the Laplace transform of $f(t)$ which we naturally replace by $F(s)$. Thus

$$\mathcal{L}\left\{\frac{\mathrm{d}f}{\mathrm{d}t}\right\} = -f(0) + sF(s)$$

As an example, we know that if $f(t) = \sin t \; u(t)$ then

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2+1} \equiv F(s)$$

and so, according to the result just obtained,

$$\begin{aligned}\mathcal{L}\left\{\frac{\mathrm{d}f}{\mathrm{d}t}\right\} = \mathcal{L}\{\cos t \; u(t)\} &= -f(0) + sF(s)\\ &= 0 + s\left(\frac{1}{s^2+1}\right)\\ &= \frac{s}{s^2+1}\end{aligned}$$

a result we know to be true.
We can find the Laplace transform of the second derivative in a similar way to find:

$$\mathcal{L}\left\{\frac{d^2f}{\mathrm{d}t^2}\right\} = -f'(0) - sf(0) + s^2F(s)$$

(The reader might wish to derive this result). Here $f'(0)$ is the derivative of $f(t)$ evaluated at $t = 0$.

Key Point

If $\quad \mathcal{L}\{f(t)\} = F(s) \quad$ then

$$\mathcal{L}\left\{\frac{\mathrm{d}f}{\mathrm{d}t}\right\} = -f(0) + sF(s)$$

$$\mathcal{L}\left\{\frac{d^2f}{\mathrm{d}t^2}\right\} = -f'(0) - sf(0) + s^2F(s)$$

If $\mathcal{L}\{f(t)\} = F(s)$ and $\dfrac{d^2 f}{dt^2} - \dfrac{df}{dt} = 3t$ with initial conditions $f(0) = 1, \quad f'(0) = 0$, find the explicit expression for $F(s)$.

Your solution

Begin by finding $\mathcal{L}\left\{\dfrac{d^2 f}{dt^2}\right\}$, $\mathcal{L}\left\{\dfrac{df}{dt}\right\}$ and $\mathcal{L}\{3t\}$

You should obtain

$$\begin{aligned} \mathcal{L}\left\{\frac{d^2 f}{dt^2}\right\} &= -f'(0) - sf(0) + s^2F(s) = -s + s^2F(s) \\ \mathcal{L}\left\{\frac{df}{dt}\right\} &= -f(0) + sF(s) = -1 + sF(s) \\ \mathcal{L}\{3t\} &= 3/s^2 \end{aligned}$$

Your solution

Now complete the calculation to find $F(s)$

You should find

$$F(s) = \frac{s^3 - s^2 + 3}{s^3(s-1)}$$

since, using the transforms we have found:

$$-s + s^2F(s) - (-1 + sF(s)) = \frac{3}{s^2}$$

so $\quad F(s)[s^2 - s] = \dfrac{3}{s^2} + s - 1 = \dfrac{s^3 - s^2 + 3}{s^2}$

leading to

$$F(s) = \frac{s^3 - s^2 + 3}{s^3(s-1)}$$

Exercises

1. Find the Laplace transforms of
(i) $t^3 e^{-2t} u(t)$ (ii) $e^t \sinh 3t.u(t)$ (iii) $\sin(t-3).u(t-3)$

2. If $F(s) = \mathcal{L}\{f(t)\}$ find expressions for $F(s)$ if
(i) $\dfrac{d^2y}{dt^2} - 3\dfrac{dy}{dt} + 4y = \sin t \qquad y(0) = 1, \quad y'(0) = 0$
(ii) $7\dfrac{dy}{dt} - 6y = 3u(t) \qquad y(0) = 0,$

3. Find the inverse Laplace transforms of
(i) $\dfrac{6}{(s+3)^4}$ (ii) $\dfrac{15}{s^2 - 2s + 10}$ (iii) $\dfrac{3s^2 + 11s + 14}{s^3 + 2s^2 - 11s - 52}$ (iv) $\dfrac{e^{-3s}}{s^4}$ (v) $\dfrac{e^{-2s-2}(s+1)}{s^2 + 2s + 5}$

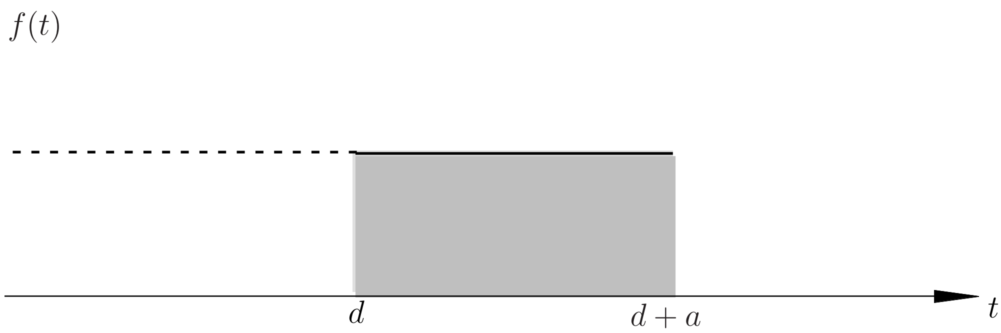
Answers 1. (i) $\dfrac{6}{(s+2)^4}$ (ii) $\dfrac{3}{(s-1)^2 - 9}$ (iii) $\dfrac{e^{-3s}}{s^2+1}$

2. (i) $\dfrac{s^3 - 3s^2 + s - 2}{(s^2+1)(s^2 - 3s + 4)}$ (ii) $\dfrac{3}{s(7s-6)}$

3. (i) $e^{-3t} t^3 u(t)$ (ii) $5e^t \sin 3t.u(t)$ (iii) $(2e^{4t} + e^{-3t}\cos 2t)u(t)$ (iv) $\frac{1}{6}(t-3)^3 u(t-3)$
(v) $e^{-t}\cos 2(t-2).u(t-2)$

3. The Delta (Impulse) Function

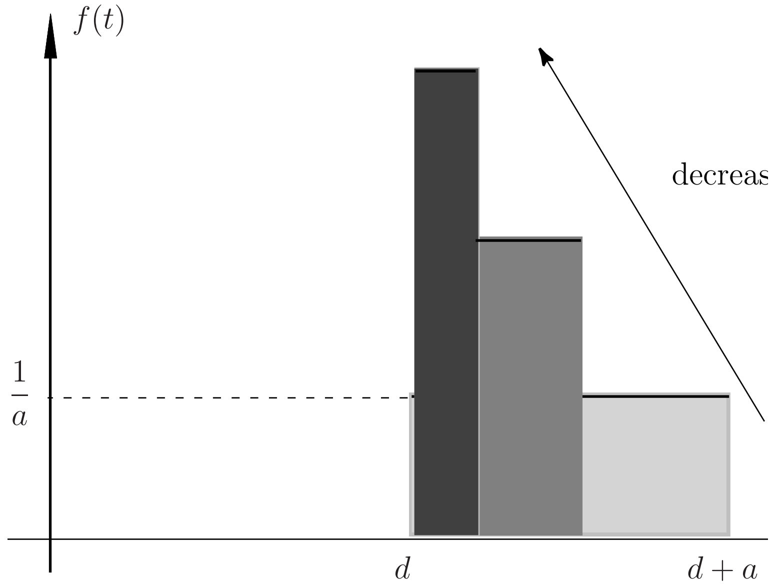
There is often a need for considering the effect on a system (modelled by a differential equation) by a forcing function which acts for a very short time interval. For example, how does the current in a circuit behave if the voltage is switched on and then very shortly afterwards switched off? How does a cantilevered beam vibrate if it is hit with a hammer (providing a certain force which acts over a very short time interval)? Both of these engineering 'systems' can be modelled by a differential equation. There are many ways the 'kick' or 'impulse' to the system can be modelled. The function we have in mind could have the graphical representation (when a is small) shown in the following figure.



This can be represented formally using step functions; it switches on at $t = d$ and switches off at $t = d + a$ and has amplitude b:

$$f(t) = b[u(t-d) - u(t - \{d+a\})]$$

The effect on the system is related to the area under the curve rather than just the amplitude b. Our aim is to reduce the time interval over which the forcing function acts (i.e. reduce a) whilst at the same time keeping the total effect (i.e. the area under the curve) a constant. To do this we shall take $b = 1/a$ so that the area is always equal to 1. Reducing the value of a then gives the sequence of inputs shown in the next figure.



As the value of a decreases the height of the rectangle increases (to ensure the value of the area under the curve is fixed at value 1) until, in the limit as $a \to 0$, the 'function' becomes a 'spike' at $t = d$. The resulting function is called a delta function (or impulse function) and denoted by $\delta(t-d)$. This notation is used because, in a very obvious sense, the delta function described here is 'located' at $t = d$. Thus the delta function $\delta(t-1)$ is 'located' at $t = 1$ whilst the delta function $\delta(t)$ is 'located' at $t = 0$.
If we were defining an ordinary function we would write

$$\delta(t-d) = \lim_{a\to 0} \frac{1}{a}[u(t-d) - u(t - \{d+a\})]$$

However, this limit does not exist. The important property of the delta function relates to its integral:

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(t-d)\mathrm{d}t &= \lim_{a\to 0} \int_{-\infty}^{\infty} \frac{1}{a}[u(t-d) - u(t-\{d+a\})]\mathrm{d}t \\ &= \lim_{a\to 0} \int_{d}^{d+a} \frac{1}{a}\mathrm{d}t \\ &= \lim_{a\to 0} \left[\frac{d+a}{a} - \frac{d}{a}\right] = 1\end{aligned}$$

which is what we expect since the area under each of the limiting curves is equal to 1.
A more technical discussion obtains the more general result:

Key Point

$$\int_{-\infty}^{\infty} f(t)\delta(t-d)\mathrm{d}t = f(d)$$

This is called the **sifting property** of the delta function as it sifts out the value $f(d)$ from the function $f(t)$. Although the integral here ranges from $t=-\infty$ to $t=+\infty$ in fact the same result is obtained for any range if the range of the integral includes the point $t=d$. That is, if $\alpha \leq d \leq \beta$ then

$$\int_{\alpha}^{\beta} f(t)\delta(t-d)\mathrm{d}t = f(d)$$

Thus, as long as the delta function is 'located' within the range of the integral the sifting property holds. For example,

$$\begin{aligned}\int_{1}^{2} \sin t\ \delta(t-1.1)\mathrm{d}t &= \sin 1.1 = 0.8112\\ \int_{0}^{\infty} \mathrm{e}^{-t}\delta(t-1)\mathrm{d}t &= \mathrm{e}^{-1} = 0.3679\end{aligned}$$

Write expressions for delta functions located at $t=-1.7$ and at $t=2.3$

Your solution

$\delta(t+1.7)$ and $\delta(t-2.3)$

Determine the integral $\int_{-1}^{3}(\sin t\ \delta(t+2) - \cos t\ \delta(t))\mathrm{d}t$

Your solution

You should obtain the value -1 since the first delta function, $\delta(t+2)$, is located outside the range of integration and thus

$$\int_{-1}^{3}(\sin t\ \delta(t+2) - \cos t\ \delta(t))\mathrm{d}t = \int_{-1}^{3} -\cos t\ \delta(t)\mathrm{d}t = -\cos 0 = -1$$

The Laplace Transform of the Delta Function

Here we consider $\mathcal{L}\{\delta(t-d)\}$. From the definition of the Laplace transform:

$$\mathcal{L}\{\delta(t-d)\} = \int_0^\infty \mathrm{e}^{-st}\delta(t-d)\mathrm{d}t = \mathrm{e}^{-sd}$$

by the sifting property of the delta function. Thus

Key Point

$$\mathcal{L}\{\delta(t-d)\} = \mathrm{e}^{-sd} \quad \text{and, putting} \quad d=0, \qquad \mathcal{L}\{\delta(t)\} = \mathrm{e}^0 = 1$$

Exercises

1. Find the Laplace transforms of
(i) $3\delta(t-3)$

Answers 1. (i) $3e^{-3s}$