Solving Differential Equations

20.4

Introduction

In this section we employ the Laplace transform to solve constant coefficient ordinary differential equations. In particular we shall consider initial value problems. We shall find that the initial conditions are automatically included as part of the solution process. The idea is simple; the Laplace transform of each term in the differential equation is taken. If the unknown function is $y(t)$ then, on taking the transform, an algebraic equation involving $Y(s) = \mathcal{L}\{y(t)\}$ is obtained. This equation is solved for $Y(s)$ which is then inverted to produce the required solution $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

Prerequisites

Before starting this Section you should . . .

① understand how to find Laplace transforms of simple functions and their derivatives

② be able to find inverse Laplace transforms using a variety of techniques

③ understand what an initial-value problem is

Learning Outcomes

After completing this Section you should be able to . . .

✓ solve initial-value problems using the Laplace transform method

1. Solving Differential Equations using the Laplace Transform

We begin with a straightforward initial value problem involving a first order constant coefficient differential equation. Let us find the solution of

$$\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = 12\mathrm{e}^{3t} \qquad y(0) = 3$$

using the Laplace transform approach.
Although it is not stated explicitly we shall assume that $y(t)$ is a causal function (we have no interest in the value of $y(t)$ if $t < 0$). Similarly, the function on the right-hand side of the differential equation ($12\mathrm{e}^{3t}$), the 'forcing function', will be assumed to be causal. (Strictly, we should write $12\mathrm{e}^{3t}u(t)$ but the step function $u(t)$ will often be omitted). Let us write $\mathcal{L}\{y(t)\} = Y(s)$. Then, taking the Laplace transform of every term in the differential equation gives:

$$\mathcal{L}\{\frac{\mathrm{d}y}{\mathrm{d}t}\} + \mathcal{L}\{2y\} = \mathcal{L}\{12\mathrm{e}^{3t}\}$$

Now

$$\begin{aligned} \mathcal{L}\{\frac{\mathrm{d}y}{\mathrm{d}t}\} &= -y(0) + sY(s) = -3 + sY(s) \\ \mathcal{L}\{2y\} &= 2Y(s) \\ \text{and} \qquad \mathcal{L}\{12\mathrm{e}^{3t}\} &= \frac{12}{s-3}. \end{aligned}$$

Substituting these expressions into the transformed version of the differential equation gives:

$$[-3 + sY(s)] + 2Y(s) = \frac{12}{s-3}$$

Solving for $Y(s)$ we have

$$(s+2)Y(s) = \frac{12}{s-3} + 3 = \frac{3+3s}{s-3}$$

Therefore

$$Y(s) = \frac{3(s+1)}{(s+2)(s-3)}$$

Now, using partial fractions, this last expression can be written in a more convenient form:

$$Y(s) = \frac{3/5}{(s+2)} + \frac{12/5}{(s-3)}$$

and then, inverting:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \tfrac{3}{5}\mathcal{L}^{-1}\{\frac{1}{s+2}\} + \tfrac{12}{5}\mathcal{L}^{-1}\{\frac{1}{s-3}\}$$

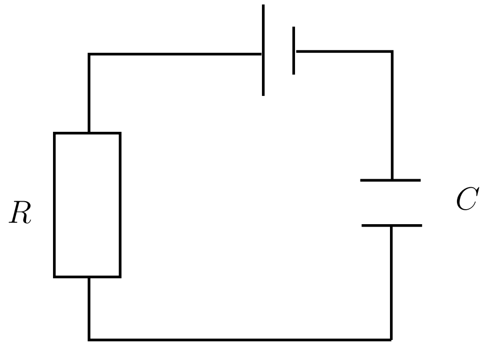
thus

$$y(t) = \tfrac{3}{5}\mathrm{e}^{-2t}u(t) + \tfrac{12}{5}\mathrm{e}^{3t}u(t)$$

This is the solution to the given initial value problem.

The equation governing the build up of charge, $q(t)$, on the capacitor of an RC circuit is

$$R\frac{dq}{dt} + \frac{1}{C}q = v_0$$



where v_0 is the constant d.c. voltage. Initially, the circuit is relaxed and the circuit 'closed' at $t = 0$ and so $q(0) = 0$ is the initial condition for the charge. Use the Laplace transform method to solve the differential equation for $q(t)$. Assume the forcing term v_0 is causal.

Your solution

Begin by finding an expression for $Q(s) = \mathcal{L}\{q(t)\}$

You should obtain $Q(s) = \dfrac{v_0 C}{s(RCs + 1)}$ since, taking the Laplace transform of each term in the differential equation:

$$R\mathcal{L}\{\frac{dq}{dt}\} + \frac{1}{C}\mathcal{L}\{q\} = \mathcal{L}\{v_0\}$$

i.e. $$R[-q(0) + sQ(s)] + \frac{1}{C}Q(s) = \frac{v_0}{s}$$

where, we emphasize, the Laplace transform of the constant term v_0 is $\frac{v_0}{s}$. Inserting $q(0) = 0$ we have, after some rearrangement,

$$Q(s) = \frac{v_0 C}{s(RCs + 1)}$$

Your solution

Now expand the expression using partial fractions

You should obtain $Q(s) = v_0C\left[\frac{1}{s} - \frac{RC}{RCs+1}\right]$

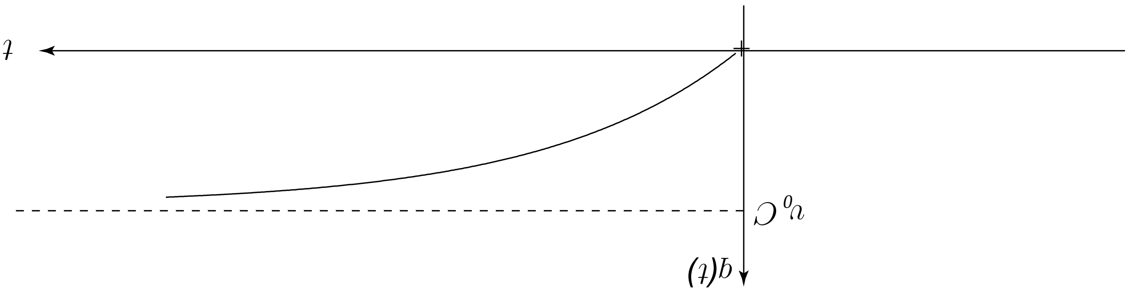
Your solution

Now obtain $q(t)$ by taking inverse Laplace transforms

You should obtain $q(t) = v_0C(1 - e^{-t/RC})u(t)$ since

$$\mathcal{L}^{-1}\{\frac{1}{s}\} = 1 \qquad \text{and} \qquad \mathcal{L}^{-1}\{\frac{RC}{RCs+1}\} = \mathcal{L}^{-1}\{\frac{1}{s+(1/RC)}\} = e^{-t/RC}$$

The solution to this problem is shown in the following diagram.



The Laplace transform method is also applied to higher-order differential equations in a similar way.

Example Solve the second-order initial-value problem:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = e^{-t} \qquad y(0) = 0 \; y'(0) = 0$$

using the Laplace transform method.

Solution

As usual we shall assume the forcing function is causal (i.e. is really $e^{-t}u(t)$). Taking the Laplace transform of each term:

$$\mathcal{L}\{\frac{d^2y}{dt^2}\} + 2\mathcal{L}\{\frac{dy}{dt}\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\}$$

that is,

$$[-y'(0) - sy(0) + s^2Y(s)] + 2[-y(0) + sY(s)] + 2Y(s) = \frac{1}{s+1}$$

Solution

Inserting the initial conditions and rearranging:

$$\begin{aligned} Y(s)[s^2+2s+2] &= \frac{1}{s+1} \\ \text{i.e.} \qquad Y(s) &= \frac{1}{(s+1)(s^2+2s+2)} \end{aligned}$$

Then, using partial fractions:

$$\begin{aligned} \frac{1}{(s+1)(s^2+2s+2)} &= \frac{1}{s+1} - \frac{(s+1)}{s^2+2s+2} \\ &= \frac{1}{s+1} - \frac{(s+1)}{(s+1)^2+1} \end{aligned}$$

where we have completed the square in the second term of the right-hand-side. We can now take the inverse Laplace transform:

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s)\} &= \mathcal{L}^{-1}\{\frac{1}{s+1}\} - \mathcal{L}^{-1}\{\frac{s+1}{(s+1)^2+1}\} \\ &= (\mathrm{e}^{-t} - \mathrm{e}^{-t}\cos t)u(t) \end{aligned}$$

which is the solution to the initial value problem.

Exercises

Use Laplace transforms to solve:

1. $\dfrac{d^2x}{\mathrm{d}t^2} + x = 2t \qquad x(0) = 0 \quad x'(0) = 5$

2. $\dfrac{\mathrm{d}x}{\mathrm{d}t} + x = 9\mathrm{e}^{2t} \qquad x(0) = 3$

Answers 1. $x(t) = 3\sin t + 2t$ 2. $x(t) = 3\mathrm{e}^{2t}$

2. Systems of Differential Equations

The Laplace transform method is also well suited to solving systems of differential equations. A simple example will illustrate the technique.

Let $x(t),\ y(t)$ be two independent functions which satisfy the coupled differential equations

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} + y &= \mathrm{e}^{-t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} - x &= 3\mathrm{e}^{-t} \\ x(0) = 0 & \qquad\qquad y(0) = 1 \end{aligned}$$

Now, using a traditional approach, we could try to eliminate one of the unknown functions from this system: for example, from the first:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\mathrm{e}^{-t} - \frac{d^2x}{\mathrm{d}t^2} \qquad \text{(taking the derivative and rearranging)}$$

This can then be substituted in the second equation: $\dfrac{\mathrm{d}y}{\mathrm{d}t} - x = 3\mathrm{e}^{-t}$, to give:

$$-\frac{d^2x}{\mathrm{d}t^2} - x = 4\mathrm{e}^{-t}$$

which can then be solved in the normal way (either using the complementary function/particular integral approach or else the Laplace transform approach). However, this approach is not workable if we have large numbers of first order differential equations to deal with. Let us instead use the Laplace transform directly.
If we use the notation that

$$\mathcal{L}\{x(t)\} = X(s) \qquad \text{and} \qquad \mathcal{L}\{y(t)\} = Y(s)$$

then, by taking the Laplace transform of every term in the given differential equations, we obtain:

$$\begin{aligned} -x(0) + sX(s) + Y(s) &= \frac{1}{s+1} \\ -y(0) + sY(s) - X(s) &= \frac{3}{s+1} \end{aligned}$$

which, using the initial conditions and rearranging gives

$$\begin{aligned} sX(s) + Y(s) &= \frac{1}{s+1} \\ -X(s) + sY(s) &= \frac{s+4}{s+1} \end{aligned}$$

Key Point

Taking the Laplace transform has transformed our system of differential equations into a system of algebraic simultaneous equations.

We can solve these algebraic equations (in $X(s)$ and $Y(s)$) using a variety of techniques (inverse matrix; Cramer's determinant method etc.) Here we will use Cramer's method.

$$X(s) = \frac{\begin{vmatrix} \frac{1}{s+1} & 1 \\ \frac{s+4}{s+1} & s \end{vmatrix}}{\begin{vmatrix} s & 1 \\ -1 & s \end{vmatrix}} = \frac{\dfrac{s}{s+1} - \dfrac{s+4}{s+1}}{s^2+1}$$

$$= \frac{-4}{(s^2+1)(s+1)} = \frac{2(s-1)}{s^2+1} - \frac{2}{s+1}$$

and

$$Y(s) = \frac{\begin{vmatrix} s & \frac{1}{s+1} \\ -1 & \frac{s+4}{s+1} \end{vmatrix}}{\begin{vmatrix} s & 1 \\ -1 & s \end{vmatrix}} = \frac{\frac{s(s+4)}{s+1} + \frac{1}{s+1}}{s^2+1}$$

$$= \frac{s^2+4s+1}{(s^2+1)(s+1)} = -\frac{1}{s+1} + \frac{2(s+1)}{s^2+1}$$

The last lines in each case having been obtained using partial fractions. We can now invert $X(s),\ Y(s)$ to find $x(t),\ y(t)$:

$$\begin{aligned}
x(t) = \mathcal{L}^{-1}\{X(s)\} &= 2\mathcal{L}^{-1}\{\frac{s}{s^2+1}\} - 2\mathcal{L}^{-1}\{\frac{1}{s^2+1}\} - 2\mathcal{L}^{-1}\{\frac{1}{s+1}\} \\
&= (2\cos t - 2\sin t - 2\mathrm{e}^{-t})u(t) \\
y(t) = \mathcal{L}^{-1}\{Y(s)\} &= -\mathcal{L}^{-1}\{\frac{1}{s+1}\} + 2\mathcal{L}^{-1}\{\frac{s}{s^2+1}\} + 2\mathcal{L}^{-1}\{\frac{1}{s^2+1}\} \\
&= (-\mathrm{e}^{-t} + 2\cos t + 2\sin t)u(t)
\end{aligned}$$

(note that once the solution for $x(t)$ is found the solution for $y(t)$ may be easier to obtain by substituting in the differential equation: $y = \mathrm{e}^{-t} - \frac{\mathrm{d}x}{\mathrm{d}t}$ rather than using Laplace transforms).

Use the Laplace transform to solve the coupled differential equations:

$$\frac{\mathrm{d}y}{\mathrm{d}t} - x = 0 \qquad \frac{\mathrm{d}x}{\mathrm{d}t} + y = 1 \qquad x(0) = -1 \quad y(0) = 1$$

Your solution

Begin by obtaining a system of algebraic equations for $X(s),\ Y(s)$

Writing $\mathcal{L}\{x(t)\} = X(s)$ and $\mathcal{L}\{y(t)\} = Y(s)$ you should obtain the set of transformed equations

$$-1 + sY(s) - X(s) = 0 \qquad 1 + sX(s) + Y(s) = \frac{1}{s}$$

which, when re-arranged, are

$$-X(s) + sY(s) = 1 \qquad sX(s) + Y(s) = \frac{1-s}{s}$$

Your solution

Now solve these equations for $X(s)$ and $Y(s)$

You should obtain

$$X(s) = -\frac{s}{1+s^2} \qquad Y(s) = \frac{1}{s} - \frac{1}{1+s^2}$$

Your solution

Now find the required solution by obtaining the inverse Laplace transforms

You should obtain $x(t) = -\cos t.u(t)$ and $y(t) = (1 - \sin t).u(t)$. This follows since

$$\begin{aligned} \mathcal{L}^{-1}\{-\frac{s}{1+s^2}\} &= -\cos t.u(t) \\ \mathcal{L}^{-1}\{\frac{1}{s}\} &= u(t) \\ \mathcal{L}^{-1}\{-\frac{1}{1+s^2}\} &= -\sin t.u(t) \end{aligned}$$

Exercises

1. Solve the given system of differential equations for the initial conditions specified.

   (a) $\frac{dx}{dt} = y \qquad \frac{dy}{dt} = x \qquad x(0) = 1 \quad y(0) = 0$

   (b) $\frac{dx}{dt} = 4x - 2y \qquad \frac{dy}{dt} = 5x + 2y \qquad x(0) = 2 \quad y(0) = -2$

2. The Laplace transform can also be used to solve a pair of coupled second order differential equations.

   Solve, for the given initial conditions,

   $$\frac{d^2x}{dt^2} = y + \sin t \qquad x(0) = 1 \quad x'(0) = 0$$

   $$\frac{d^2y}{dt^2} = -\frac{dx}{dt} + \cos t \qquad y(0) = -1 \quad y'(0) = -1$$

   (Note that the initial conditions on each of $x(t)$ and $y(t)$ are needed in the second order situation.)

1. (a) $x = \cosh t \quad y = \sinh t$ (b) $x = e^{3t}(2\cos 3t + 2\sin 3t) \quad y = e^{3t}(-2\cos 3t + 4\sin 3t)$

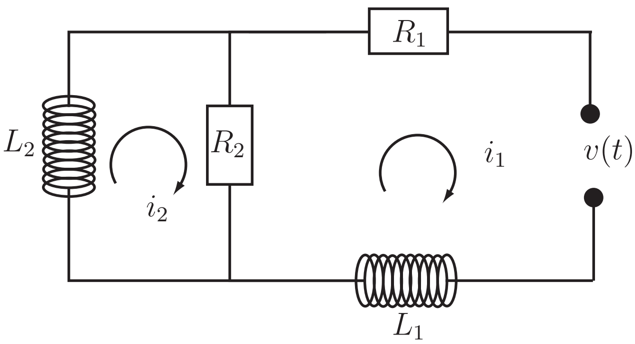
2. $x = \cos t \quad y = -\cos t - \sin t$

3. Applications of Systems of Differential Equations

Coupled electrical circuits and mechanical vibrating systems involving several masses in springs offer examples of engineering systems modelled by systems of differential equations.

Electrical circuits

Consider the RL (resistance/inductance) circuit with a voltage $v(t)$ applied as shown.



If i_1 and i_2 denote the currents in each loop we obtain, using Kirchhoff's voltage law:

(i) in the right loop:

$$L_1\frac{\mathrm{d}i_1}{\mathrm{d}t} + R_1(i_1 - i_2) + R_2 i_1 = v(t)$$

(ii) in the left loop:

$$L_2\frac{\mathrm{d}i_2}{\mathrm{d}t} + R_1(i_2 - i_1) = 0$$

Suppose, in the above circuit, that

$L_1 = 0.8$ henry $\quad L_2 = 1$ henry $\quad R_1 = 1\ \Omega \quad R_2 = 1.4\ \Omega$.

Assume zero initial conditions: $i_1(0) = i_2(0) = 0$. Suppose that the applied voltage is constant:

$$v(t) = 100 \text{ volts} \qquad t \geq 0.$$

We shall solve the problem by Laplace transforms. Begin by obtaining $V(s)$, the Laplace transform of $v(t)$.

Your solution

We have, from the definition of the Laplace transform:

$$V(s) = \int_0^\infty 100e^{-st}\,dt = 100\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{100}{s}$$

This is simply the Laplace transform of the step function of height 100.

Now insert the parameter values into the differential equations and obtain the Laplace transform of each equation. Denote by $I_1(s)$, $I_2(s)$ the Laplace transforms of the unknown currents. (These are equivalent to $X(s)$ and $Y(s)$ of the theory.)

Your solution

$$0.8\frac{\mathrm{d}i_1}{\mathrm{d}t} + i_1 - i_2 + 1.4i_1 = v(t)$$

$$\frac{\mathrm{d}i_2}{\mathrm{d}t} + i_2 - i_1 = 0$$

Rearranging and dividing the first equation by 0.8:

$$\frac{\mathrm{d}i_1}{\mathrm{d}t} + 3i_1 - 1.25i_2 = 1.25v(t)$$

$$\frac{\mathrm{d}i_2}{\mathrm{d}t} - i_1 + i_2 = 0$$

Taking Laplace transforms and inserting the (zero value) initial conditions:

$$(s+3)I_1(s) - 1.25I_2(s) = \frac{125}{s}$$

$$-I_1(s) + (s+1)I_2(s) = 0$$

Now solve these equations for $I_1(s)$ and $I_2(s)$, put each expression into partial fractions and finally take the inverse Laplace transform to obtain $i_1(t)$ and $i_2(t)$.

Your solution

We find

$$I_1(s) = \frac{125(s+1)}{s(s+1/2)(s+7/2)} = \frac{500}{7s} - \frac{125}{3(s+1/2)} - \frac{625}{21(s+7/2)}$$

in partial fractions.

Hence $\quad i_1(t) = \dfrac{500}{7} - \dfrac{125}{3}\mathrm{e}^{-t/2} - \dfrac{625}{21}\mathrm{e}^{-7t/2}$

Similarly

$$I_2(s) = \frac{125}{s(s+1/2)(s+7/2)} = \frac{500}{7s} - \frac{250}{3(s+1/2)} + \frac{250}{21(s+7/2)}$$

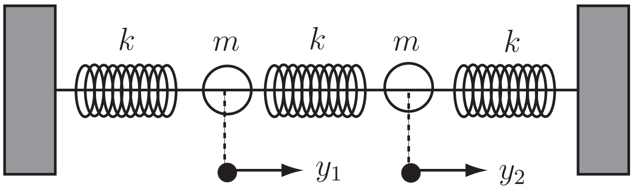
which has inverse Laplace transform:

$$i_2(t) = \frac{500}{7} - \frac{250}{3}\mathrm{e}^{-t/2} + \frac{250}{21}\mathrm{e}^{-7t/2}$$

Notice in both cases that $i_1(t)$ and $i_2(t)$ will tend to the steady state value $\dfrac{500}{7}$ as t increases.

Two masses on springs

Considerthe vibrating system shown:



As you can see, the system consists of two equal masses, both m, and 3 springs of the same stiffness k. The governing differential equations can be obtained by applying Newton's Second Law ('force equals mass times acceleration'): (recall that a single spring of stiffness k will experience a force $-ky$ if it is displaced a distance y from its equilibrium).

In our system therefore

$$m\frac{\mathrm{d}^2 y_1}{\mathrm{d}t^2} = -ky_1 + k(y_2 - y_1)$$

$$m\frac{\mathrm{d}^2 y_2}{\mathrm{d}t^2} = -k(y_2 - y_1) - ky_2$$

which is a **pair** of second order differential equations.

If $m = 1$, $k = 2$ and the initial conditions are

$$y_1(0) = 1 \quad y_1'(0) = \sqrt{6} \quad y_2(0) = 1 \quad y_2'(0) = -\sqrt{6}$$

solve, using Laplace transforms the system of differential equations to find $y_1(t)$ and $y_2(t)$.

Begin by letting $Y_1(s), Y_2(s)$ be the Laplace transforms of $y_1(t), y_2(t)$ respectively and take the transforms of the differential equations, inserting the initial conditions.

Your solution

$$s^2Y_1(s) - s - \sqrt{6} = -4Y_1(s) + 2Y_2(s)$$

$$s^2Y_2(s) - s + \sqrt{6} = 2Y_1(s) - 4Y_2(s)$$

or

$$(s^2+4)Y_1 - 2Y_2 = s + \sqrt{6}$$

$$-2Y_1 + (s^2+4)Y_2 = s - \sqrt{6}$$

Solve these equations, by Cramer's rule or by elimination, then use partial fractions and finally take inverse Laplace transforms.

Your solution

$$Y_1(s) = \frac{(s+\sqrt{6})(s^2+4) + 2(s-\sqrt{6})}{(s^2+4)^2 - 4} = \frac{s}{s^2+2} + \frac{\sqrt{6}}{s^2+6}$$

from which

$$y_1(t) = \cos\sqrt{2}t + \sin\sqrt{6}t$$

A similar calculation, which you should perform, gives

$$y_2(t) = \cos\sqrt{2}t - \sin\sqrt{6}t$$

We see that the motion of each mass is composed of two harmonic oscillations; the system model was undamped so, on this model, the vibration continues indefinitely.