The Convolution Theorem

20.5

Introduction

In this section we introduce the convolution of two functions $f(t)$, $g(t)$ which we denote by $(f * g)(t)$. The convolution is an important construct because of the Convolution Theorem which gives the inverse Laplace transform of a product of two transformed functions:

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$$

Prerequisites

Before starting this Section you should . . .

① be able to find Laplace and inverse Laplace transforms of simple functions

② be able to integrate by parts

③ understand how to use step functions in integration

Learning Outcomes

After completing this Section you should be able to . . .

✓ calculate the convolution of simple functions

✓ apply the Convolution Theorem to obtain inverse Laplace transforms

1. The Convolution

Let $f(t)$ and $g(t)$ be two functions of t. The convolution of $f(t)$ and $g(t)$ is also a function of t, denoted by $(f * g)(t)$ and is defined by the relation

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-x)g(x)\mathrm{d}x$$

However if f and g are both *causal* functions then (strictly) $f(t), g(t)$ are written $f(t)u(t)$ and $g(t)u(t)$ respectively, so that

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-x)u(t-x)g(x)u(x)\mathrm{d}x = \int_{0}^{t} f(t-x)g(x)\mathrm{d}x$$

because of the properties of the step functions ($u(t-x) = 0$ if $x > t$ and $u(x) = 0$ if $x < 0$).

Key Point

If $f(t)$ and $g(t)$ are causal functions then their convolution is defined by:

$$(f * g)(t) = \int_{0}^{t} f(t-x)g(x)\mathrm{d}x$$

This is an odd looking definition but it turns out to have considerable use both in Laplace transform theory and in the modelling of linear engineering systems. The reader should note that the variable of integration is x, as far as the integration process is concerned the t-variable is (temporarily) regarded as a constant.

Example Find the convolution of f and g if $f(t) = tu(t)$ and $g(t) = t^2u(t)$.

Solution

$$f(t-x) = (t-x)u(t-x) \quad \text{and} \quad g(x) = x^2u(x)$$

Therefore

$$\begin{aligned}(f * g)(t) = \int_{0}^{t} (t-x)x^2\mathrm{d}x &= \left[\tfrac{1}{3}x^3t - \tfrac{1}{4}x^4\right]_0^t \\ &= \tfrac{1}{3}t^4 - \tfrac{1}{4}t^4 = \tfrac{1}{12}t^4\end{aligned}$$

Example Find the convolution of $f(t) = tu(t)$ and $g(t) = \sin t.u(t)$.

Solution

Here $f(t-x) = (t-x)u(t-x)$ and $g(x) = \sin x.u(x)$ and so

$$(f * g)(t) = \int_0^t (t-x)\sin x \mathrm{d}x$$

We will need to integrate by parts. We find, remembering again that t is a constant in the integration process,

$$\begin{aligned}\int_0^t (t-x)\sin x \mathrm{d}x &= \Big[-(t-x)\cos x\Big]_0^t - \int_0^t (-1)(-\cos x)\mathrm{d}x \\ &= [0+t] - \int_0^t \cos x \mathrm{d}x \\ &= t - \Big[\sin x\Big]_0^t = t - \sin t\end{aligned}$$

so that

$$(f * g)(t) = t - \sin t \quad \text{or, equivalently, in this case} \quad (t * \sin t)(t) = t - \sin t$$

In the last example we found the convolution of $f(t) = tu(t)$ and $g(t) = \sin t.u(t)$. In this exercise you are asked to find the convolution $(g * f)(t)$ that is, to reverse the order of f and g.

Your solution

Begin by writing $(g * f)(t)$ as an appropriate integral

You should find, according to the definition $g(t-x) = \sin(t-x).u(t-x)$ and $f(x) = xu(x)$ and so

$$(g * f)(t) = \int_0^t \sin(t-x).x\mathrm{d}x$$

Your solution

Now evaluate the convolution integral

You should find $(g*f)(t) = t - \sin x$ since

$$(g*f)(t) = \int_0^t \sin(t-x)\cdot x\,dx$$
$$= \left[x\cos(t-x)\right]_0^t - \int_0^t \cos(t-x)dx$$
$$= [t-0] + [\sin(t-x)]_0^t = t - \sin t$$

This exercise illustrates the general result

Key Point

$$(f*g)(t) = (g*f)(t)$$

in words: the convolution of $f(t)$ with $g(t)$ is the same as the convolution of $g(t)$ with $f(t)$.

Obtain the Laplace transforms of $f(t) = tu(t)$ and $g(t) = \sin t.u(t)$ and of $(f*g)(t)$. What do you notice?

Your solution

Begin by finding $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{g(t)\}$

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2} \qquad \mathcal{L}\{g(t)\} = \frac{1}{s^2+1} \qquad \text{from tables}$$

Your solution

Now find $\mathcal{L}\{(f * g)(t)\}$

From the example above $(f * g)(t) = t - \sin t$ and so

$$\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{t - \sin t\} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

Your solution

What do you observe?

$$\mathcal{L}\{(f * g)(t)\} = \frac{1}{s^2} - \frac{1}{s^2+1} = \frac{1}{s^2}\left(\frac{1}{s^2+1}\right) = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$$

We see that the Laplace transform of the convolution of $f(t)$ and $g(t)$ is the product of their separate Laplace transforms. This, in fact, is a general result which is expressed in the statement of the *convolution theorem* which we discuss in the next section

2. The Convolution Theorem

Let $f(t)$ and $g(t)$ be causal functions with Laplace transforms $F(s)$ and $G(s)$ respectively, i.e. $\mathcal{L}\{f(t)\} = F(s)$ adn $\mathcal{L}\{g(t)\} = G(s)$. Then it can be shown that

Key Point

$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$ or equivalently $\mathcal{L}\{(f * g)(t)\} = F(s)G(s)$

Example Use the Convolution Theorem to find the inverse transform of $\dfrac{6}{s(s^2+9)}$.

Solution

In this case we can, of course, find the inverse transform by using partial fractions and then using the table of transforms. That is:

$$\frac{6}{s(s^2+9)} = \frac{(2/3)}{s} - \frac{(2/3)s}{s^2+9}$$

and so

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{6}{s(s^2+9)}\right\} &= \tfrac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \tfrac{2}{3}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} \\ &= \tfrac{2}{3}u(t) - \tfrac{2}{3}\cos 3t.u(t)\end{aligned}$$

However, we can alternatively use the Convolution Theorem. Let us choose

$$F(s) = \frac{2}{s} \qquad \text{and} \qquad G(s) = \frac{3}{s^2+9}$$

then

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = 2u(t) \quad \text{and} \quad g(t) = \mathcal{L}^{-1}\{G(s)\} = \sin 3t.u(t)$$

So

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)G(s)\} &= (f*g)(t) \qquad \text{by the Convolution Theorem} \\ &= \int_0^t 2u(t-x)\sin 3x.u(x)\mathrm{d}x\end{aligned}$$

Now the variable t can take any value from $-\infty$ to $+\infty$. If $t<0$ then the variable of integration, x, is negative and so $u(x)=0$. We conclude that

$$(f*g)(t) = 0 \quad \text{if} \quad t<0$$

that is, $(f*g)(t)$ is a **causal function**. Let us now consider the other possibility for t, that is the range $t \geq 0$. Now, in the range of integration $0 \leq x \leq t$ and so

$$u(t-x) = 1 \qquad u(x) = 1$$

since both $t-x$ and x are non-negative. Therefore

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)G(s)\} &= \int_0^t 2\sin 3x\mathrm{d}x \\ &= \left[-\tfrac{2}{3}\cos 3x\right]_0^t = -\tfrac{2}{3}(\cos 3t - 1) \qquad t \geq 0\end{aligned}$$

Hence

$$\mathcal{L}^{-1}\{\frac{6}{s(s^2+9)}\} = -\tfrac{2}{3}(\cos 3t - 1)u(t)$$

which agrees with the value obtained above using the partial fraction approach.

Use the Convolution Theorem to find the inverse transform of $H(s) = \dfrac{s}{(s-1)(s^2+1)}$.

Your solution

Begin by choosing two functions of s: that is, $F(s)$ and $G(s)$

Although there are many possibilities it would seem sensible to choose

$$F(s) = \frac{1}{s-1} \qquad \text{and} \qquad G(s) = \frac{s}{s^2+1}$$

since, by inspection, we can write down their inverse Laplace transforms:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = e^t u(t) \qquad \text{and} \qquad g(t) = \mathcal{L}^{-1}\{G(s)\} = \cos t.u(t)$$

Your solution

Now construct the convolution integral

You should obtain

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}\{H(s)\} \\ &= \mathcal{L}^{-1}\{F(s)G(s)\} \\ &= \int_0^t f(t-x)g(x)\,dx = \int_0^t e^{t-x}u(t-x)\cos x.u(x)\,dx \end{aligned}$$

Your solution

Now complete the evaluation of the integral. (Treat the cases $t < 0$ and $t \geq 0$ separately.)

You should find $h(t) = \frac{1}{2}(\sin t - \cos t + e^t)u(t)$ since $h(t) = 0$ if $t < 0$ and

$$\begin{aligned} h(t) &= \int_0^t e^{t-x}\cos x\,dx \qquad \text{if} \quad t \geq 0 \\ &= \Big[e^{t-x}\sin x\Big]_0^t - \int_0^t (-1)e^{t-x}\sin x\,dx \qquad \text{by parts} \\ &= \sin t + \Big[-e^{t-x}\cos x\Big]_0^t - \int_0^t (-e^{t-x})(-\cos x)dx \\ &= \sin t - \cos t + e^t - h(t) \end{aligned}$$

$$\text{or} \qquad 2h(t) = \sin t - \cos t + e^t \qquad t \geq 0$$

Finally $h(t) = \frac{1}{2}(\sin t - \cos t + e^t)u(t)$

Exercises

1. Find the convolution of (i) $2tu(t)$ and $t^3u(t)$ (ii) $e^t u(t)$ and $tu(t)$ (iii) $e^{-2t}u(t)$ and $e^{-t}u(t)$.
In each case reverse the order to check that $(f * g)(t) = (g * f)(t)$.

2. Use the Convolution Theorem to determine the inverse Laplace transforms of
(i) $\dfrac{1}{s^2(s+1)}$ (ii) $\dfrac{1}{(s-1)(s-2)}$ (iii) $\dfrac{1}{(s^2+1)^2}$

Answers 1. (i) $\frac{1}{10}t^5$ (ii) $-t - 1 + e^t$ (iii) $e^{-t} - e^{-2t}$

2. (i) $(t - 1 + e^{-t})u(t)$ (ii) $(-e^t + e^{2t})u(t)$ (iii) $\frac{1}{2}(\sin t - t\cos t)u(t)$