Applications of Eigenvalues and Eigenvectors

22.2

Introduction

Many applications of matrices in both engineering and science utilize eigenvalues and, sometimes, eigenvectors. Control theory, vibration analysis, electric circuits, advanced dynamics and quantum mechanics are just a few of the application areas.

Many of the applications involve the use of eigenvalues and eigenvectors in the process of **transforming** a given matrix into a **diagonal** matrix and we discuss this process in this Section. We then go on to show how this process is invaluable in solving coupled differential equations of both first order and second order.

Prerequisites

Before starting this Section you should . . .

① have a knowledge of determinants and matrices

② have a knowledge of linear first order differential equations

Learning Outcomes

After completing this Section you should be able to . . .

✓ diagonalize a matrix with distinct eigenvalues using the modal matrix

✓ solve systems of linear differential equations by the 'decoupling' method

1. Applications of Eigenvalues and Eigenvectors

Diagonalization of a Matrix with distinct eigenvalues

Diagonalization means transforming a non-diagonal matrix into an equivalent matrix which is diagonal and hence is simpler to deal with.

A matrix A with distinct eigenvalues has, as we mentioned in property 3 in Section 22.1, eigenvectors which are linearly independent. If we form a matrix P whose columns are these eigenvectors, it can then be shown that

$$\det(P) \neq 0$$

so that P^{-1} exists.

The product $P^{-1}AP$ is then a **diagonal** matrix D whose diagonal elements are the eigenvalues of A. Thus if $\lambda_1, \lambda_2, \ldots \lambda_n$ are the distinct eigenvalues of A with associated eigenvectors $X^{(1)}, X^{(2)}, \ldots, X^{(n)}$ respectively:

$$P = \left[\begin{array}{ccccccc} X^{(1)} & \vdots & X^{(2)} & \vdots & \cdots & \vdots & X^{(n)} \\ & & & & & & \end{array} \right]$$

will produce a product

$$P^{-1}AP = D = \left[\begin{array}{cccc} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & & & \\ 0 & \ldots & \ldots & \lambda_n \end{array} \right]$$

We see that the order of the eigenvalues in D matches the order in which P is formed from the eigenvectors.
N.B.

(a) The matrix P is called the **modal matrix** of A

(b) Since D, as a diagonal matrix, has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ which are the same as those of A then the matrices D and A are said to be **similar**. The transformation of A into D using

$$P^{-1}AP = D$$

is said to be a **similarity transformation**

Example Let $A = \left[\begin{array}{cc} 2 & 3 \\ 3 & 2 \end{array} \right]$. Obtain the modal matrix P and calculate the product $P^{-1}AP$. (The eigenvalues and eigenvectors of this particular matrix A were obtained earlier in this workbook).

Solution

The matrix A has two distinct eigenvalues $\lambda_1 = -1$, $\lambda_2 = 5$ with corresponding eigenvectors $X_1 = \begin{bmatrix} x \\ -x \end{bmatrix}$ and $X_2 = \begin{bmatrix} x \\ x \end{bmatrix}$. We can therefore form the modal matrix from the simplest eigenvectors of these forms:

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

(Other eigenvectors would be acceptable e.g. we could use $P = \begin{bmatrix} 2 & 3 \\ -2 & 3 \end{bmatrix}$ but there is no reason to over complicate the calculation).

It is easy to obtain the inverse of this 2×2 matrix P and the reader should confirm that:

$$P^{-1} = \frac{1}{\det(P)}\,\text{adj}(P) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^T = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

We can now construct the product $P^{-1}AP$:

$$\begin{aligned} \therefore \qquad P^{-1}AP &= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 5 \\ 1 & 5 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} -2 & 0 \\ 0 & 10 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \end{aligned}$$

as expected. Show(by repeating the method outlined above) that had we defined $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ (i.e. interchanged the order in which the eigenvectors were taken) we would find $P^{-1}AP = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$ (i.e. the resulting diagonal elements would also be interchanged).

The matrix $A = \begin{bmatrix} -1 & 4 \\ 0 & 3 \end{bmatrix}$ has eigenvalues -1 and 3 and associated eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively.

If $P_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$, $P_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

write down the products $P_1^{-1}AP_1$, $P_2^{-1}AP_2$, $P_3^{-1}AP_3$

(You may not need to do detailed calculations).

Your solution

$$P_1^{-1}AP_1 = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} = D_1$$

$$P_2^{-1}AP_2 = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} = D_2$$

$$P_3^{-1}AP_3 = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = D_3$$

Note that $D_1 = D_2$, demonstrating that any eigenvectors of A can be used to form P. Note also that since the columns of P_1 have been interchanged in forming P_3 then so have the eigenvalues in D_3 as compared with D_1.

Matrix Powers

If $P^{-1}AP = D$ then we can obtain A as the subject of this matrix equation as follows:

multiply on the left by P and on the right by P^{-1} to obtain

$$PP^{-1}APP^{-1} = PDP^{-1}$$

But $\quad PP^{-1} = P^{-1}P = I$

$$\therefore \qquad IAI = PDP^{-1} \qquad \text{and so} \qquad A = PDP^{-1}$$

We can use this result to obtain the **powers** of a square matrix, a process which is sometimes useful in control theory. Note that

$$A^2 = A.A \qquad A^3 = A.A.A. \quad \text{etc.}$$

as we would expect: clearly obtaining high powers of A directly would involve many multiplications. The process is quite straightforward, however, for a diagonal matrix D.

Obtain D^2 and D^3 if $D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$. Write down D^{10}.

Your solution

$$D^2 = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 3^2 & 0 \\ 0 & (-2)^2 \end{bmatrix}$$

$$D^3 = \begin{bmatrix} 3^2 & 0 \\ 0 & (-2)^2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & (-2) \end{bmatrix} = \begin{bmatrix} 3^3 & 0 \\ 0 & (-2)^3 \end{bmatrix}$$

Continuing in this way: $D^{10} = \begin{bmatrix} 3^{10} & 0 \\ 0 & (-2)^{10} \end{bmatrix}$

We now use the relation

$$A = PDP^{-1}$$

to obtain a formula for powers of A in terms of the easily calculated powers of the diagonal matrix D

$$A^2 = A.A = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PDIDP^{-1} = PD^2P^{-1}$$

Similarly:

$$A^3 = A^2.A = (PD^2P^{-1})(PDP^{-1}) = PD^2(P^{-1}P)DP^{-1} = PD^3P^{-1}$$

or, in general,

$$\boxed{A^k = PD^kP^{-1}}$$

Example If $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ find A^{23}.

(Use the results of the worked example).

Solution

We know from the previous worked example that if $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$$P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} = D$$

where $P^{-1} = \dfrac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$\begin{aligned} \therefore \quad A &= PDP^{-1} \\ \therefore \quad A^{23} &= PD^{23}P^{-1} \qquad \text{using the general result shown above} \\ \text{i.e.} \quad A &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 5^{23} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

which is easily determined.

Exercises

1. Find a diagonalising matrix P if

(a) $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & -2 & 3 \end{bmatrix}$

Verify, in each case, that $P^{-1}AP$ is diagonal, with the eigenvalues of A as its diagonal elements.

Answers

1. (a) $P = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$ (b) $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}$

Systems of First Order Differential Equations

Systems of first order ordinary differential equations arise in many areas of mathematics and engineering, for example in control theory and in the analysis of electrical circuits. In each case the basic unknowns are each a function of the time variable t. A number of techniques have been developed to solve such systems of equations; for example the Laplace transform or the use of the exponential matrix (outside the scope of this discussion). Here we shall use eigenvalues and eigenvectors to obtain the solution. Our first step will be to recast the system of ordinary differential equations in the *matrix form* $\dot{X} = AX$ where A is an $n \times n$ coefficient matrix of constants, X is the $n \times 1$ column vector of unknown functions and $\dot{X}$ is the $n \times 1$ column vector containing the derivatives of the unknowns.. The main step will be to use the modal matrix of A to diagonalise the system of differential equations. This process will transform $\dot{X} = AX$ into the form $\dot{Y} = DY$ where D is a *diagonal* matrix. We shall find that this new diagonal system of differential equations can be easily solved. This special solution will allow us to obtain the solution of the original system.

Obtain the solutions of the pair of first order differential equations

$$\left.\begin{aligned} \dot{x} &= -2x \\ \dot{y} &= -5y \end{aligned}\right\} \qquad (*)$$

given the **initial conditions**

$$\begin{aligned} x(0) &= 3 \qquad \text{i.e. } x = 3 \text{ at } t = 0 \\ y(0) &= 2 \qquad \text{i.e. } y = 2 \text{ at } t = 0 \end{aligned}$$

(The notation is that $\dot{x} = \dfrac{dx}{dt}$, $\dot{y} = \dfrac{dy}{dt}$)

Recall, from your course on differential equations, that the general solution of the differential equation $\dfrac{dy}{dt} = Ky$ is $y = y_0 e^{Kt}$.

Your solution

Using the hint: $x = x_0 e^{-2t}$ $\quad y = y_0 e^{-5t}$ $\quad$ where $x_0 = x(0)$ and $y_0 = y(0)$.

From the given initial condition $x_0 = 3$ $\quad y_0 = 2$ $\quad$ so finally $x = 3e^{-2t}$ $\quad y = 2e^{-5t}$.

In the above example although we had two differential equations to solve they were really quite separate. We needed no knowledge of matrix theory to solve them.

However, we should note that the two differential equations here can be written in matrix form.

Thus if $X = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \qquad A = \begin{bmatrix} -2 & 0 \\ 0 & -5 \end{bmatrix}$

the 2 equations (*) can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

i.e. $\dot{X} = AX$.

Write the pair of **coupled** differential equations

$$\left.\begin{aligned} \dot{x} &= 4x + 2y \\ \dot{y} &= -x + y \end{aligned}\right\} \qquad (**)$$

in matrix form.

Your solution

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\dot{X} = AX$$

The essential difference between the two pairs of differential equations just considered is that the first pair $(*)$ were really separate equations, the first equation of $(*)$ involving only the unknown x, the second involving only y. In matrix terms this corresponded to a **diagonal** matrix A in the system $\dot{X} = AX$. The second system $(**)$ of equations were coupled in that

both equations involved **both** x and y. This corresponded to the **non-diagonal** matrix A in the system $\dot{X} = AX$.

Clearly the second system here is more difficult to deal with than the first and there is where we can use our knowledge of diagonalisation.

Example Find the solution of the *coupled* differential equations

$$\begin{aligned} \dot{x} &= 4x + 2y \\ \dot{y} &= -x + y \\ \text{with initial conditions} \qquad x(0) &= 1 \qquad y(0) = 0 \end{aligned}$$

Here $\dot{x} \equiv \dfrac{\mathrm{d}x}{\mathrm{d}t}$ and $\dot{y} \equiv \dfrac{\mathrm{d}y}{\mathrm{d}t}$.

Solution

Defining as above

$$X = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \qquad \text{and} \qquad \dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}.$$

the original system of differential equations can be written, as we have seen,

$$\dot{X} = AX \qquad \text{where} \qquad A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \text{ in the present example.}$$

We now introduce a new column vector of unknowns $Y = \begin{bmatrix} r(t) \\ s(t) \end{bmatrix}$ through the relation

$$X = PY$$

where P is the modal matrix of A. Then, since P is a matrix of constants:

$$\begin{aligned} \dot{X} &= P\dot{Y} \\ \text{so} \quad \dot{X} = AX \quad \text{becomes} \quad P\dot{Y} &= AX = A(PY) \\ \text{Then, multiplying by } P^{-1} \text{ on the left,} \\ \dot{Y} &= (P^{-1}AP)Y \end{aligned}$$

But, because of the properties of the modal matrix, we know that $P^{-1}AP$ is a **diagonal matrix**. Thus if λ_1, λ_2 are **distinct** eigenvalues of A then:

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Hence $\dot{Y} = (P^{-1}AP)Y$ becomes

$$\begin{bmatrix} \dot{r} \\ \dot{s} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}.$$

Solution (contd.)

That is, when written out we have

$$\begin{aligned}\dot{r} &= \lambda_1 r\\ \dot{s} &= \lambda_2 s.\end{aligned}$$

These equations are **de-coupled**. The first equation only involves the unknown function $r(t)$ and has solution $r(t) = Ce^{\lambda_1 t}$. The second equation only involves the unknown function $s(t)$ and has solution $s(t) = Ke^{\lambda_2 t}$ where C, K are arbitrary constants.
Once r, s are known the original unknowns x, y can be found from the relation $X = PY$.

Note that the theory outlined above is applicable to any system of differential equations of the form

$$\dot{X} = AX$$

where A is an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Consider the present example in which

$$A = \begin{bmatrix} 4 & 2\\ -1 & 1\end{bmatrix}.$$

It is easily checked that A has distinct eigenvalues $\lambda_1 = 3$ $\lambda_2 = 2$ and corresponding eigenvectors $X_1 = \begin{bmatrix} -2\\ 1\end{bmatrix}$, $X_2 = \begin{bmatrix} 1\\ -1\end{bmatrix}$. Therefore, if

$$P = \begin{bmatrix} -2 & 1\\ 1 & -1\end{bmatrix} \qquad \text{then} \qquad P^{-1}AP = \begin{bmatrix} 3 & 0\\ 0 & 2\end{bmatrix}$$

and (from above),

$$r(t) = Ce^{3t} \qquad s(t) = Ke^{2t}.$$

So

$$\begin{aligned}\begin{bmatrix} x\\ y\end{bmatrix} \equiv X = PY &= \begin{bmatrix} -2 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix} r\\ s\end{bmatrix}\\ &= \begin{bmatrix} -2 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix} Ce^{3t}\\ Ke^{2t}\end{bmatrix}\\ &= \begin{bmatrix} -2Ce^{3t} + Ke^{2t}\\ Ce^{3t} - Ke^{2t}\end{bmatrix}.\end{aligned}$$

Solution (contd.)

Therefore

$$\begin{aligned} x &= -2Ce^{3t} + Ke^{2t} \\ y &= Ce^{3t} - Ke^{2t}. \end{aligned}$$

We can now impose the initial conditions $x(0) = 1$ and $y(0) = 0$ to give

$$\begin{aligned} 1 &= -2C + K \\ 0 &= C - K. \end{aligned}$$

Thus $C = K = -1$ and the solution to the original system of differential equations is

$$\begin{aligned} x(t) &= 2e^{3t} - e^{2t} \\ y(t) &= -e^{3t} + e^{2t}. \end{aligned}$$

The approach we have demonstrated in this example can be extended to

(a) Systems of first order differential equations containing more than 2 unknowns

(b) systems of second order differential equations

The only restriction, as we have said, is that the matrix A in the system $\dot{X} = AX$ has distinct eigenvalues.

Systems of second order differential equations

The 'decoupling method' discussed above can be readily extended to this situation which could arise, for example, in a mechanical system consisting of coupled springs.
A typical example of such a system with two unknowns has the form

$$\ddot{x} = ax + by$$

$$\ddot{y} = cx + dy$$

or, in matrix form,

$$\ddot{X} = AX$$

where $X = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\quad \ddot{x} = \frac{d^2x}{dt^2}$, $\quad \ddot{y} = \frac{d^2y}{dt^2}$

Make the substitution $X = PY$ where $Y = \begin{bmatrix} r(t) \\ s(t) \end{bmatrix}$ and P is the modal matrix of A, A being assumed here to have distinct eigenvalues λ_1 and λ_2. Solve the resulting pair of 'decoupled' equations for the case, which arises in practice, where λ_1 and λ_2 are both negative.

Your solution

Exactly as with a first order system putting $X = PY$ into the second order system $\ddot{X} = AX$ gives

$$\ddot{Y} = P^{-1}APY \quad \text{that is} \quad \ddot{Y} = DY \quad \text{where} \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

In full

$$\begin{bmatrix} \ddot{r} \\ \ddot{s} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$$

That is, $\ddot{r} = \lambda_1 r = -\omega_1^2 r$ and $\ddot{s} = \lambda_2 s = -\omega_2^2 s$

(for the case where λ_1 and λ_2 are both negative.) The two uncoupled equations are of the form of the differential equation governing simple harmonic motion. Hence the general solutions are

$$\begin{aligned} r &= A\cos\omega_1 t + B\sin\omega_1 t \\ s &= C\cos\omega_2 t + E\sin\omega_2 t \end{aligned}$$

The solutions for x and y are then obtained by use of

$$X = PY.$$

Note that in this second order case four initial conditions, two each for both x and y, are required because four constants A, B, C, E arise in the solution.

Exercises

1. Solve by decoupling each of the following systems:

   (a) $\dfrac{dX}{dt} = AX$ where $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}, \quad X(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

   (b) $\dot{x}_1 = x_2$

   $\dot{x}_2 = x_1 + 3x_3$

   $\dot{x}_3 = x_2$

   with $x_1(0) = 2, \quad x_2(0) = 0, \quad x_3(0) = 2$

   (c) $\dfrac{dX}{dt} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} X$ with $X(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

   (d) $\dot{x}_1 = x_1$

   $\dot{x}_2 = -2x_2 + x_3$

   $\dot{x}_3 = 4x_2 + x_3$

   with $x_1(0) = x_2(0) = x_3(0) = 1$

2. Matrix methods can also be used as we have discussed to solve systems of **second**-order differential equations such as might arise with coupled electrical or mechanical systems. For example the motion of two masses m_1 and m_2 vibrating on coupled springs, neglecting damping and spring masses, is governed by

$$\begin{aligned} m_1\ddot{y}_1 &= -k_1y_1 + k_2(y_2 - y_1) \\ m_2\ddot{y}_2 &= -k_2(y_2 - y_1) \end{aligned}$$

   where dots denote derivatives with respect to time. Write this system as a matrix equation $\ddot{Y} = AY$ and use the decoupling method to find Y if

   (i) $m_1 = m_2 = 1, \; k_1 = 3, \; k_2 = 2$

   and the initial conditions are $y_1(0) = 1$, $y_2(0) = 2$, $\dot{y}(0) = -2\sqrt{6}$, $\dot{y}_2(0) = \sqrt{6}$

   (ii) $m_1 = m_2 = 1, \quad k_1 = 6, \; k_2 = 4$

   and the initial conditions are $y_1(0) = y_2(0) = 0, \quad \dot{y}_1(0) = \sqrt{2}, \quad \dot{y}_2(0) = 2\sqrt{2}$

   Verify your solutions by substitution in each case.

Answers

1. (a) $X = \begin{bmatrix} 2e^{5t} & - & e^{-5t} \\ e^{5t} & + & 2e^{-5t} \end{bmatrix}$ (b) $X = \begin{bmatrix} 2 & \cosh 2t \\ 4 & \sinh 2t \\ 2 & \cosh 2t \end{bmatrix}$

   (c) $X = \dfrac{1}{4}\begin{bmatrix} e^{5t} & + & 3e^{t} \\ e^{5t} & - & e^{t} \\ e^{5t} & - & e^{t} \end{bmatrix}$ (d) $X = \dfrac{1}{5}\begin{bmatrix} 5e^{t} & & \\ 2e^{2t} & + & 3e^{-3t} \\ 8e^{2t} & - & 3e^{-3t} \end{bmatrix}$

2. (i) $Y = \begin{bmatrix} \cos t - 2\sin\sqrt{6}t \\ 2\cos t + \sin\sqrt{6}t \end{bmatrix}$ (ii) $Y = \begin{bmatrix} \sin\sqrt{2}t \\ 2\sin\sqrt{2}t \end{bmatrix}$