

# Repeated Eigenvalues and Symmetric Matrices

22.3



## Introduction

In this Section we further develop the theory of eigenvalues and eigenvectors in two distinct directions. Firstly we look at matrices where one or more of the eigenvalues is **repeated**. We shall see that this sometimes (but not always) causes problems in the diagonalization process that was discussed in the previous Section. Rather more pleasantly, perhaps, we shall then consider the special properties possessed by **symmetric** matrices which make them particularly easy to work with.



## Prerequisites

Before starting this Section you should ...

- ① have a knowledge of determinants and matrices
- ② have a knowledge of linear first order differential equations



## Learning Outcomes

After completing this Section you should be able to ...

- ✓ state the conditions under which a matrix with repeated eigenvalues may be diagonalized
- ✓ state the main properties of real symmetric matrices

# 1. Matrices with repeated eigenvalues

So far we have considered the diagonalization of matrices with distinct (i.e. non-repeated) eigenvalues. We have accomplished this by the use of a **non-singular** modal matrix  $P$  i.e. one where  $\det P \neq 0$  and hence the inverse  $P^{-1}$  exists. We now want to discuss briefly the case of a matrix  $A$  with at least one pair of repeated eigenvalues. We shall see that for some such matrices diagonalization **is** possible but for others it is not.

The crucial question is whether we can form a non-singular modal matrix  $P$  with the eigenvectors of  $A$  as columns.

## Example

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$$

which has characteristic equation

$$\det(A - \lambda I) = (1 - \lambda)(1 - \lambda) = 0$$

So the only eigenvalue is 1 which is repeated or, more formally, has **multiplicity 2**.

To obtain eigenvectors of  $A$  corresponding to  $\lambda = 1$  we proceed as usual and solve

$$AX = 1X$$

or

$$\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

implying

$$x = x \quad \text{and} \quad -4x + y = y$$

from which  $x = 0$  and  $y$  is arbitrary.

Thus possible eigenvectors are

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \quad \dots$$

However if we attempt to form a modal matrix  $P$  from any two of these eigenvectors, e.g.

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

then the resulting matrix

$$P = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$$

has **zero** determinant. Thus  $P^{-1}$  does **not** exist and the similarity transformation  $P^{-1}AP$  that we have used previously to diagonalize a matrix is not possible here.

The essential point, at a slightly deeper level, is that the columns of  $P$  in this case are not linearly independent since

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

i.e. one is a simple multiple of the other.

This situation is to be contrasted with that of a matrix with non-repeated eigenvalues. Earlier, for example, we showed that the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

has the non-repeated eigenvalues  $\lambda_1 = -1$ ,  $\lambda_2 = 5$  with associated eigenvectors

$$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

These two eigenvectors **are** linearly independent.

since  $\begin{bmatrix} 1 \\ -1 \end{bmatrix} \neq k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  for any value of  $k \neq 0$ .

Then the modal matrix

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

has linearly independent columns: so that  $\det P \neq 0$  and  $P^{-1}$  exists.

The general result, illustrated by this second example, is given in the following keypoint.



### Key Point

Eigenvectors corresponding to distinct eigenvalues are always linearly independent.

It follows from this that we can **always** diagonalize an  $n \times n$  matrix with  $n$  **distinct** eigenvalues since it will possess  $n$  linearly independent eigenvectors. We can then use these as the columns of  $P$  secure in the knowledge that these columns will be linearly independent and hence  $P^{-1}$  will exist.

It follows, in considering the case of repeated eigenvalues, that the key problem is whether or not there are still  $n$  linearly independent eigenvectors for an  $n \times n$  matrix.

We shall now consider two  $3 \times 3$  examples as illustrations.



Let 
$$A = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Obtain the eigenvalues and eigenvectors of  $A$ . Can we obtain three linearly independent eigenvectors for  $A$ ? Can  $A$  be diagonalized?

**Your solution**

The conclusion is that since  $A$  is  $3 \times 3$  and we can only obtain **two** linearly independent eigenvectors then  $A$  **cannot be diagonalized**.

$X$  and  $Y$  are certainly linearly independent (as we would expect since they correspond to distinct eigenvalues.) However, there is only one independent eigenvector of the form  $Y$  corresponding to the repeated eigenvalue  $-2$ .

where  $\beta \neq 0$  is arbitrary.

$$Y = \begin{bmatrix} 0 \\ \beta \\ -3\beta \end{bmatrix} = \beta \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

from which  $z = 0$ ,  $x + 3y = 0$  so the eigenvectors are all of the form

$$\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2x \\ -2y \\ -2z \end{bmatrix}$$

For the repeated eigenvalue  $\lambda = -2$  we must solve  $AY = (-2)Y$  for the eigenvector  $Y$ :

arbitrary,  $\alpha \neq 0$ .

from which  $x = z = 0$  and  $y$  is arbitrary. Thus an eigenvector is  $X = \begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix}$  where  $\alpha$  is

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

For  $\lambda = 1$  the associated eigenvectors satisfy

i.e.  $(-2 - \lambda)(1 - \lambda)(-2 - \lambda) = 0$  which gives  $\lambda = 1$ ,  $\lambda = -2$ ,  $\lambda = -2$ .

$$\det(A - \lambda I) = \begin{vmatrix} -2 - \lambda & 0 & 0 \\ 1 & 1 - \lambda & 0 \\ 0 & 0 & -2 - \lambda \end{vmatrix} = 0$$

The characteristic equation of  $A$  is



The matrix  $A = \begin{bmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{bmatrix}$  has eigenvalues  $-3, 1, 1$ . The eigen-

vector corresponding to the eigenvalue  $-3$  is  $X = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$  or any multiple.

Investigate carefully the eigenvectors associated with the repeated eigenvalue  $\lambda = 1$  and deduce whether  $A$  can be diagonalized.

### Your solution

$$D = P^{-1}AP = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We could then indeed diagonalize  $A$  through the transformation

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

We can thus form a non-singular modal matrix  $P$  from  $Y_1$  and  $Y_2$  together with  $X$

$$Y_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

All three equations here give  $x - y + z = 0$ . Here we have just one equation in three unknowns so we can choose any two values arbitrarily. The choices  $x = 1, y = 0$  (and hence  $z = -1$ ) and  $x = 0, y = 1$  (and hence  $z = 1$ ) for example, give rise to linearly independent eigenvectors

$$\text{i.e.} \quad \begin{bmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

We must solve  $AY = (1)Y$  for the required eigenvector



## Key Point

An  $n \times n$  matrix with repeated eigenvalues can be diagonalized provided we can obtain  $n$  linearly independent eigenvectors for it. This will be the case if, for each repeated eigenvalue  $\lambda_i$ , of multiplicity  $m_i > 1$ , we can obtain  $m_i$  linearly independent eigenvectors.

## 2. Symmetric matrices

Symmetric matrices have a number of useful properties which we will investigate in this Section.



Consider the following pairs of matrices

$$A_1 = \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix} \quad A_2 = \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 5 & 8 & 7 \\ -1 & 6 & 8 \\ 3 & 4 & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 5 & 8 & 7 \\ 8 & 6 & 4 \\ 7 & 4 & 0 \end{bmatrix}$$

What property do the matrices  $A_2$  and  $B_2$  possess that  $A_1$  and  $B_1$  do not?

### Your solution

Matrices  $A_2$  and  $B_2$  are **symmetric** across the principal diagonal. In other words transposing these matrices, i.e. interchanging their rows and columns, does not change them.

$$A_2^T = A_2 \quad B_2^T = B_2.$$

This property does not hold for matrices  $A_1$  and  $B_1$  which are **non-symmetric**.

Now calculating the eigenvalues of an  $n \times n$  matrix with real elements involves, in principle at least, solving an  $n$ 'th order polynomial equation, a quadratic equation if  $n = 2$ , a cubic equation if  $n = 3$  and so on. As is well known such equations sometimes have only real solutions but complex solutions (occurring as complex conjugate pairs) can also arise. This situation can therefore arise with the eigenvalues of matrices.

For example consider the non-symmetric matrix

$$A = \begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix}$$



Obtain the eigenvalues of  $A$  and show that they form a complex conjugate pair.

**Your solution**

giving eigenvalues  $\pm i$  which are of course complex conjugates.

$$-(2 - \lambda)(2 + \lambda) + 5 = 0 \quad \text{leading to} \quad \lambda^2 + 1 = 0$$

i.e.

$$\det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & 5 \\ -1 & -2 - \lambda \end{vmatrix} = 0$$

The characteristic equation of  $A$  is

In particular any  $2 \times 2$  matrix of the form

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

has complex conjugate eigenvalues  $a \pm ib$ .

A  $3 \times 3$  example of a matrix with some complex eigenvalues is

$$B = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

A straightforward calculation shows that the eigenvalues of  $B$  are

$$\lambda = -1 \text{ (real), } \pm i \text{ (complex conjugates).}$$

With **symmetric** matrices on the other hand, complex eigenvalues are not possible.



**Key Point**

The eigenvalues of a symmetric matrix with real elements are always real.

The general proof of this result is beyond our scope but a simple proof for symmetric  $2 \times 2$  matrices is straightforward.

Let  $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$  be any  $2 \times 2$  symmetric matrix,  $a, b, c$  being real numbers.

The characteristic equation for  $A$  is

$$(a - \lambda)(c - \lambda) - b^2 = 0 \quad \text{or, expanding} \quad \lambda^2 - (a + c)\lambda + ac - b^2 = 0$$

from which

$$\lambda = \frac{(a + c) \pm \sqrt{(a + c)^2 - 4ac + 4b^2}}{2}$$

The quantity under the square root sign can be treated as follows:

$$a^2 + c^2 + 2ac - 4ac + b^2 = (a - c)^2 + 4b^2$$

which is always positive and hence  $\lambda$  cannot be complex.



Obtain the eigenvalues and also the eigenvectors of the symmetric  $2 \times 2$  matrix

$$A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

### Your solution

The characteristic equation for  $A$  is  $(4 - \lambda)(1 - \lambda) + 4 = 0$  or  $\lambda^2 - 5\lambda = 0$  giving  $\lambda = 0$  and  $\lambda = 5$ , both of which are of course real and also unequal (or distinct). For the larger eigenvalue  $\lambda = 5$  the eigenvectors  $X = \begin{bmatrix} x \\ y \end{bmatrix}$  satisfies  $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5x \\ 5y \end{bmatrix}$  i.e.  $-x - 2y = 0$  or  $-2x - 4y = 0$  Both equations tell us that  $x = -2y$  so an eigenvector for  $\lambda = 5$  is  $X = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$  or any multiple of this. For  $\lambda = 0$  the associated eigenvectors satisfy  $4x - 2y = 0$  or  $-2x + y = 0$  i.e.  $y = 2x$  (from both equations) so an eigenvector is  $Y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  or any multiple.



Let us now look more closely at the eigenvectors  $X$  and  $Y$  in the above example. In particular we consider the product  $X^T Y$ .



Evaluate  $X^T Y$  from the previous guided exercise i.e. where

$$X = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad Y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

**Your solution**

When two  $n \times 1$  column vectors (of any size  $n$ ) possess this property i.e.  $X^T Y = 0$  they are said to be **orthogonal**.

$$0 = X^T Y = (2 \quad -1) \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2 \times 1 - 1 \times 2 = 2 - 2 = 0$$



### Key Point

Two  $n \times 1$  column vectors  $X$  and  $Y$  are orthogonal if  $X^T Y = 0$ .



We obtained earlier in the workbook the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

which, as we now emphasize, is symmetric. We found that the eigenvalues were  $2$ ,  $2 + \sqrt{2}$ ,  $2 - \sqrt{2}$  which are real (and distinct). The corresponding eigenvectors were, respectively

$$X = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad Y = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} \quad Z = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

(or, as usual, any multiple of these).

Show that these three eigenvectors  $X, Y, Z$  are **mutually orthogonal**.

## Your solution

hence verifying the mutual orthogonality of these three eigenvectors.

$$0 = 1 - 1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} (1 \quad \sqrt{2} \quad 1) = X^T Z$$

$$0 = 1 + 2 - 1 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} (1 \quad \sqrt{2} \quad -1) = Z^T Y$$

$$0 = 1 - 1 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} (1 \quad 0 \quad 1) = Y^T X$$

## General Theory

The following proof that eigenvectors corresponding to distinct eigenvalues of a symmetric matrix are orthogonal is straightforward and you are encouraged to follow it through.

Let  $A$  be a symmetric  $n \times n$  matrix and let  $\lambda_1, \lambda_2$  be two distinct eigenvalues of  $A$  i.e.  $\lambda_1 \neq \lambda_2$  with associated eigenvectors  $X, Y$  respectively. We have seen that  $\lambda_1$  and  $\lambda_2$  must be real since  $A$  is symmetric. Then

$$AX = \lambda_1 X \quad AY = \lambda_2 Y$$

Transposing the first of these results gives

$$X^T A^T = \lambda_1 X^T \tag{1}$$

(Remember that for any two matrices the transpose of a product is the product of the transposes **in reverse order**.)

We now multiply both sides of (1) on the right by  $Y$  (as well as putting  $A^T = A$ , since  $A$  is symmetric) to give:

$$X^T AY = \lambda_1 X^T Y \tag{2}$$

But using the second eigenvalue equation (2) becomes

$$X^T \lambda_2 Y = \lambda_1 X^T Y$$

or, since  $\lambda_2$  is just a number,

$$\lambda_2 X^T Y = \lambda_1 X^T Y$$

Taking all terms to the same side and factoring gives

$$(\lambda_2 - \lambda_1)X^TY = 0$$

from which, since by assumption  $\lambda_1 \neq \lambda_2$ , we obtain the result

$$X^TY = 0$$

and the orthogonality has been proved.



### Key Point

The eigenvectors associated with distinct eigenvalues of a symmetric matrix are mutually orthogonal.

The reader familiar with the ordinary algebra of vectors will recall that for two vectors whose Cartesian forms are

$$\underline{a} = a_x\underline{i} + a_y\underline{j} + a_z\underline{k} \quad \underline{b} = b_x\underline{i} + b_y\underline{j} + b_z\underline{k}$$

the scalar (or dot) product is

$$\underline{a} \cdot \underline{b} = a_x b_x + a_y b_y + a_z b_z.$$

Furthermore, if  $\underline{a}$  and  $\underline{b}$  are mutually perpendicular then  $\underline{a} \cdot \underline{b} = 0$ .

(The word ‘orthogonal’ is sometimes used instead of perpendicular in the case).

Our result that two column vectors are orthogonal if  $X^TY = 0$  may thus be considered as a generalisation of the 3-dimensional result  $\underline{a} \cdot \underline{b} = 0$ .

## Diagonalization of symmetric matrices

Recall from our earlier work that

1. We can **always** diagonalize a matrix with distinct eigenvalues (whether these are real or complex).
2. We can **sometimes** diagonalize a matrix with repeated eigenvalues. (The condition for this to be possible was that any eigenvalue of multiplicity  $m$  had to have associated with it  $m$  linearly independent eigenvectors).

The situation with symmetric matrices is simpler. Basically we can diagonalize **any** symmetric matrix.

To take the discussions further we first need the concept of an **orthogonal** matrix.

A square matrix  $A$  is said to be orthogonal if its inverse (if it exists) is equal to its transpose:

$$A^{-1} = A^T \quad \text{or, equivalently,} \quad AA^T = A^T A = I.$$

**Example** An important example of an orthogonal matrix is

$$A = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix}$$

(which arises when we use matrices to describe rotations in a plane).  
Clearly

$$\begin{aligned} AA^T &= \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \\ &= \begin{bmatrix} \cos^2 \phi + \sin^2 \phi & 0 \\ 0 & \sin^2 \phi + \cos^2 \phi \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \end{aligned}$$

It is clear that  $A^T A = I$  also, so  $A$  is indeed orthogonal.

It can be shown, but we omit the details, that any  $2 \times 2$  matrix which is orthogonal can be written in one of the two forms.

$$\begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}$$

If we look closely at either of these matrices we can see that

1. the two columns are mutually orthogonal e.g. for the first matrix we have

$$(\cos \phi \quad -\sin \phi) \begin{bmatrix} \sin \phi \\ \cos \phi \end{bmatrix} = \cos \phi \sin \phi - \sin \phi \cos \phi = 0$$

2. each column has **magnitude** 1 (because  $\sqrt{\cos^2 \phi + \sin^2 \phi} = \sqrt{1}$ )

Although we shall not prove it these results are necessary and sufficient for any order square matrix to be orthogonal.



### Key Point

A matrix is orthogonal if and only if

1. its columns are mutually orthogonal
2. each column has unit magnitude



For each of the following matrices verify that the two properties above are satisfied. Then check in both cases that  $AA^T = A^T A = I$  i.e. that  $A^T = A^{-1}$ .

$$(a) \quad A = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \quad (b) \quad A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

**Your solution**

(a)  $A = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

Since  $\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$  and  $\begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$  are orthogonal.

Since  $\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{1} = 1$  and  $\sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$  each column has unit magnitude.

Straightforward multiplication shows  $AA^T = A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ .

(b) Proceed as in (i).

The following is the key result of this Section.



### Key Point

**Any** symmetric matrix  $A$  can be diagonalized using an orthogonal modal matrix  $P$  via the transformation

$$P^T A P = D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

The proof of this result is omitted but it follows that any  $n \times n$  symmetric matrix **must** possess  $n$  mutually orthogonal eigenvectors **even if some of the eigenvalues are repeated**.

It should be clear to the reader that this is a very powerful result for any applications that involve diagonalization of a symmetric matrix. Further if we do need to find the inverse of  $P$  (this is unnecessary in some applications) then this is a trivial process since

$$P^{-1} = P^T.$$



Consider the symmetric matrix

$$A = \begin{bmatrix} 1 & 0 & \sqrt{2} \\ 0 & 2 & 0 \\ \sqrt{2} & 0 & 0 \end{bmatrix}$$

This has eigenvalues 2, 2,  $-1$  i.e. the eigenvalue 2 is repeated with multiplicity 2.

Associated with the non-repeated eigenvalue  $-1$  is an eigenvector

$$X = \begin{bmatrix} 1 \\ 0 \\ -\sqrt{2} \end{bmatrix}$$

or any multiple.

- i. Normalize the eigenvector  $X$
- ii. Investigate the eigenvectors associated with the repeated eigenvalue 2.

**Your solution**

Here both  $\alpha$  and  $\beta$  are arbitrary. (The equations (\*) give us no information about  $y$  which is why its value is arbitrary.)

$$Y = \begin{bmatrix} \beta \\ \alpha \\ \sqrt{2}\beta \end{bmatrix}$$

Then  $Y$  has the form

$$\text{i.e. } x = \sqrt{2}z$$

$$\begin{aligned} 0 &= -x + \sqrt{2}z \\ 0 &= \sqrt{2}x - z \end{aligned}$$

The first and third equations give

$$(*) \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} z \\ y \\ x \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & -2 \\ 0 & 0 & 0 \\ \sqrt{2} & -1 & 0 \end{bmatrix}$$

which gives

$$AY = 2Y$$

ii. The eigenvectors associated with  $\lambda = 2$  satisfy

$$\begin{bmatrix} \frac{\sqrt{3}}{1} \\ 1 \\ 0 \\ -\sqrt{\frac{3}{2}} \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ 0 \\ 1 \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

i. Normalizing  $X$  which has magnitude  $\sqrt{1^2 + (-\sqrt{2})^2} = \sqrt{3}$  gives

A certain amount of care is now required in the choice of  $\alpha$  and  $\beta$  if we are to find an orthogonal modal matrix to diagonalize  $A$ .

For any choice

$$X^T Y = (1 \quad 0 \quad -\sqrt{2}) \begin{bmatrix} \sqrt{2}\beta \\ \alpha \\ \beta \end{bmatrix} = \sqrt{2}\beta - \sqrt{2}\beta = 0$$

So  $X$  and  $Y$  are orthogonal. (The normalization of  $X$  does not affect this).

However we also need two **orthogonal** eigenvectors of the form  $Y$ :

two such are

$$Y^{(1)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad (\text{choosing } \beta = 0, \alpha = 1)$$



$$Y^{(2)} = \begin{bmatrix} \sqrt{2} \\ 0 \\ 1 \end{bmatrix} \quad (\text{choosing } \alpha = 0, \beta = 1)$$

After normalization, these become

$$Y^{(1)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad Y^{(2)} = \begin{bmatrix} \sqrt{\frac{2}{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Hence the matrix

$$P = \begin{bmatrix} X & \vdots & Y^{(1)} & \vdots & Y^{(2)} \end{bmatrix} \\ = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \sqrt{\frac{2}{3}} \\ 0 & 1 & 0 \\ -\sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$$

is orthogonal and diagonalizes  $A$ :

$$P^T A P = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

## Hermitian matrices

In some applications, of which quantum mechanics is one, matrices with **complex** elements arise.

If  $A$  is such a matrix then the matrix  $\overline{A}^T$  is the **conjugate transpose** of  $A$  i.e. the complex conjugate of each element of  $A$  is taken as well as  $A$  being transposed. Thus if

$$A = \begin{bmatrix} 2+i & 2 \\ 3i & 5-2i \end{bmatrix} \quad (*)$$

$$\overline{A}^T = \begin{bmatrix} 2-i & -3i \\ 2 & 5+2i \end{bmatrix}$$

An **Hermitian** matrix is one satisfying

$$\overline{A}^T = A$$

The matrix (\*) above is clearly non-Hermitian. Indeed the most obvious features of an Hermitian matrix is that its diagonal elements **must** be real. (Can you see why?) Thus

$$A = \begin{bmatrix} 6 & 4+i \\ 4-i & -2 \end{bmatrix}$$

is Hermitian.

A  $3 \times 3$  example of an Hermitian matrix is

$$A = \begin{bmatrix} 1 & i & 5-2i \\ -i & 3 & 0 \\ 5+2i & 0 & 2 \end{bmatrix}$$

An Hermitian matrix is in fact a generalization of a symmetric matrix. The key property of an Hermitian matrix is the same as that of a real symmetric matrix – i.e. the eigenvalues are always **real**.