Contents 23

Fourier **series**

Learning **outcomes**

needs doing

Time **allocation**

You are expected to spend approximately thirteen hours of independent study on the material presented in this workbook. However, depending upon your ability to concentrate and on your previous experience with certain mathematical topics this time may vary considerably.

Periodic Functions

23.1

Introduction

You should already know how to take a function of a single variable $f(x)$ and represent it by a power series in x about any point x_0 of interest. Such a series is known as a Taylor series or Taylor expansion or, if $x_0 = 0$, as a Maclaurin series. This expansion is only possible if the function is sufficiently smooth (that is, if it can be differentiated as often as required). Geometrically this means that there are no *jumps* or *spikes* in the curve $y = f(x)$ near the point of expansion. However, in many practical situations the functions we have to deal with are not as well behaved as this and so no power series expansion in x is possible. Nevertheless, if the function is **periodic**, so that it repeats over and over again, then, irrespective of the function's behaviour, (that is, no matter how many *jumps* or *spikes* it has) the function may be expressed as a series of sines and cosines. Such a series is called a **Fourier series**.

Fourier series have many applications in mathematics, in physics and in engineering. For example they are sometimes essential in solving problems (in heat conduction, wave propagation etc) that involve partial differential equations. Also, using Fourier series the analysis of many engineering systems (such as electric circuits or mechanical vibrating systems) can be extended from the case where the input to the system is a sinusoidal function to the more general case where the input is periodic but non-sinsusoidal.

Prerequisites

Before starting this Section you should . . .

① be familiar with trigonometric functions

Learning Outcomes

After completing this Section you should be able to . . .

✓ recognise periodic functions

✓ be able to determine the frequency, the amplitude and the period of a sinusoid

✓ be able to represent common periodic functions by trigonometric Fourier series.

1. Introduction

You have met in earlier Mathematics courses the concept of representing a function by an infinite series of simpler functions such as polynomials. For example, the Maclaurin series representing e^x has the form

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

or, in the more concise sigma notation,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

(remembering that 0! is defined as 1).

The basic idea is that for those values of x for which the series converges we may approximate the function by using only the first few terms of the series.

Fourier series, which we discuss in this and the following Sections, are also usually infinite series but involve sine and cosine functions (or their complex exponential equivalents) rather than polynomials. They are widely used for approximating **periodic functions**. Such approximations are of considerable use in science and engineering. For example, elementary a.c. theory provides techniques for analyzing electrical circuits when the currents and voltages present are assumed to be sinusoidal. Fourier Series enable us to extend such techniques to the situation where the functions (or signals) involved are periodic but not actually sinusoidal. You may also see in Workbook 25 that Fourier series sometimes have to be used when solving partial differential equations.

2. Periodic Functions

A function $f(t)$ is periodic if the function values repeat at regular intervals of the independent variable t. The regular interval is referred to as the **period**. See Figure 1.

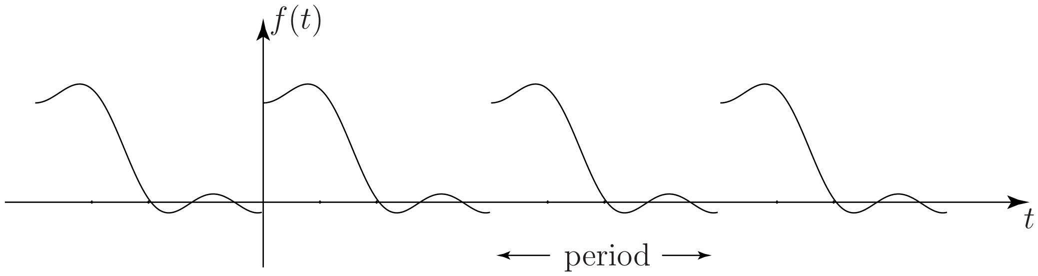

Figure 1

If P denotes the period we have

$$f(t + P) = f(t)$$

for any value of t.

The most obvious examples of periodic functions are the trigonometric functions $\sin t$ and $\cos t$, both of which have period 2π (using radian measure as we shall do throughout this unit.) This follows since

$$\sin(t + 2\pi) = \sin t \qquad \text{and} \qquad \cos(t + 2\pi) = \cos t$$

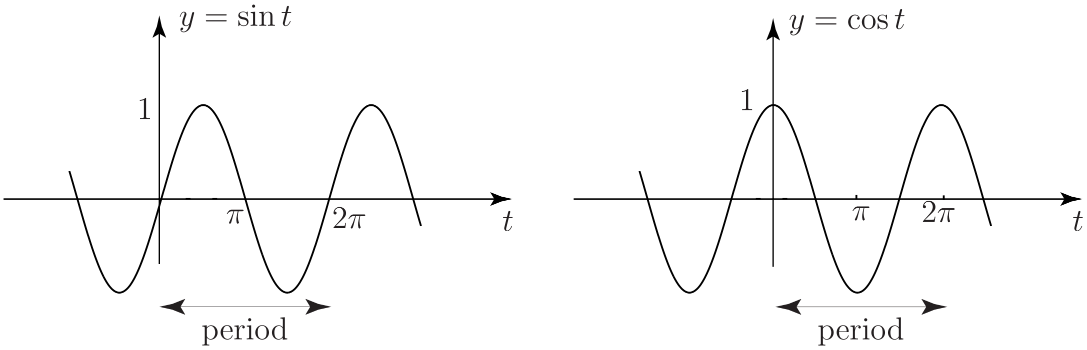

Figure 2

The **amplitude** of these sinusoidal functions is the maximum displacement from $y = 0$ and is clearly 1. (Note that we use the term sinusoidal to include cosine as well as sine functions.)

More generally we can consider a sinusoid

$$y = A \sin nt$$

which has maximum value, or amplitude, A and where n is usually a positive integer.

For example

$$y = \sin 2t$$

is a sinusoid of amplitude 1 and period $\frac{2\pi}{2} = \pi$. The fact that the period is π follows because

$$\sin 2(t + \pi) = \sin(2t + 2\pi) = \sin 2t$$

for any value of t.

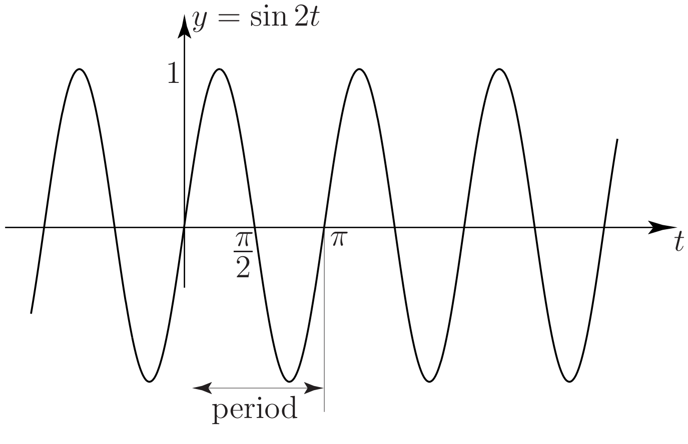

Figure 3

We see that $y = \sin 2t$ has half the period of $\sin t$ (π as opposed to 2π). This can alternatively be phrased by stating that $\sin 2t$ oscillates twice as rapidly (or has twice the **frequency**) of $\sin t$.

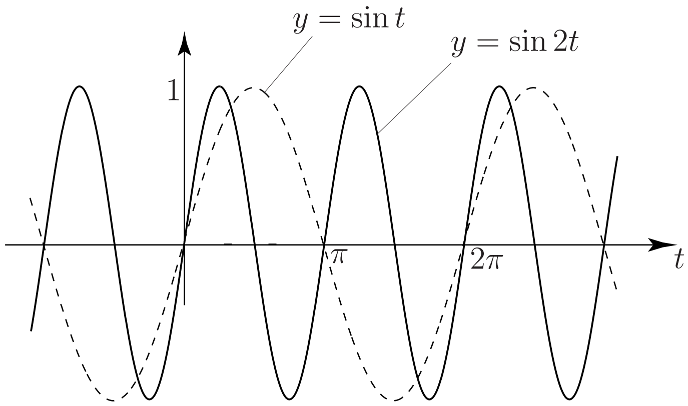

Figure 4

In general

$$y = A \sin nt$$

has amplitude A, period $\frac{2\pi}{n}$ and completes n oscillations when t changes by 2π.
Formally, we define the **frequency** of a sinusoid as the reciprocal of the period:

$$\text{frequency} = \frac{1}{\text{period}}$$

and the angular frequency (often denoted the Greek Letter ω (omega)) as

$$\begin{aligned} \text{angular frequency} &= 2\pi \times \text{ frequency} \\ &= \frac{2\pi}{\text{period}} \end{aligned}$$

Thus

$$y = A \sin nt$$

has frequency $\frac{n}{2\pi}$ and angular frequency n.

State the amplitude, period, frequency and angular frequency of
(i) $y = 5 \cos 4t$ (ii) $y = 6 \sin \frac{2t}{3}$.

Your solution

For (i) we have

amplitude 5, period $\frac{2\pi}{4} = \frac{\pi}{2}$, frequency $\frac{2}{\pi}$, angular frequency 4

Your solution

For (ii) we have

amplitude 6, period 3π, frequency $\frac{1}{3\pi}$, angular frequency $\frac{2}{3}$

Harmonics

In representing a non-sinusoidal function of period 2π by a Fourier Series we shall see shortly that only certain sinusoids will be required:

(a) $A_1 \cos t$ (and $B_1 \sin t$)

These also have period 2π and together are referred to as the first (or fundamental) harmonic.

(b) $A_2 \cos 2t$ (and $B_2 \sin 2t$)

These have half the period, or double the frequency of the first harmonic and are referred to as the second harmonic.

(c) $A_3 \cos 3t$ (and $B_3 \sin 3t$)

These have period $\frac{2\pi}{3}$ and constitute the third harmonic.

In general the Fourier Series of a function of period 2π will require harmonics of the type

$$A_n \cos nt \qquad (\text{and} \;\; B_n \sin nt) \qquad \text{where} \quad n = 1, 2, 3, \ldots$$

Non-sinusoidal periodic functions.

The following are examples of such functions (they are often called "waves"):

Square Wave

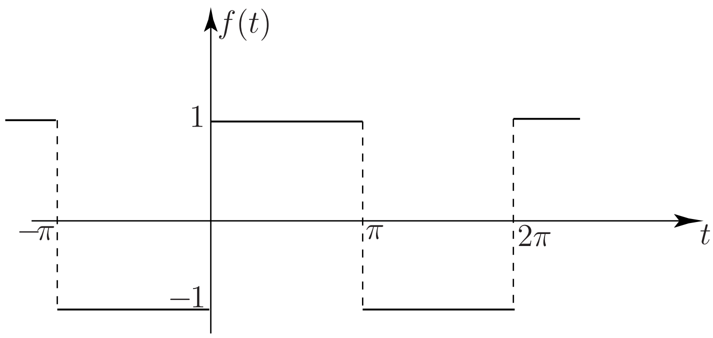

Figure 5

Analytically we can describe this function as follows:

$$f(t) = \begin{cases} -1 & -\pi < t < 0 \\ +1 & \;\;\, 0 < t < \pi \end{cases}$$

(which gives the definition over one period.)

$$f(t + 2\pi) = f(t)$$

(which tells us that the function has period 2π).

Saw-tooth wave

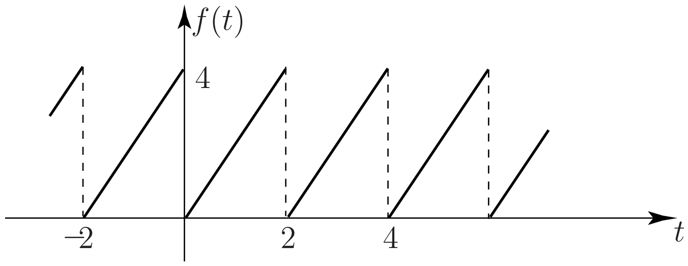

Figure 6

In this case we can describe the function as follows:

$$\begin{aligned} f(t) &= 2t \qquad\qquad 0 < t < 2 \\ f(t+2) &= f(t) \end{aligned}$$

Here the period is 2, the frequency is $\frac{1}{2}$ and the angular frequency is $\frac{2\pi}{2} = \pi$.

Triangular wave

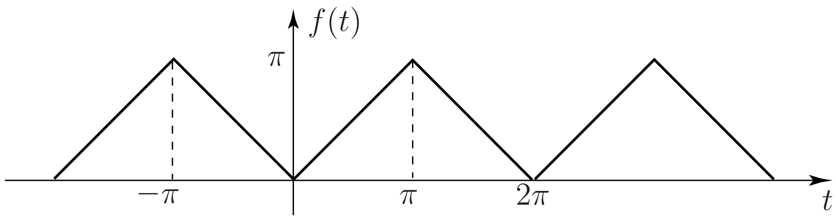

Figure 7

Here we can conveniently define the function using $-\pi < t < \pi$ as the "basic period":

$$f(t) = \begin{cases} -t & -\pi < t < 0 \\ t & 0 < t < \pi \end{cases}$$

or, more concisely,

$$f(t) = |t| \qquad -\pi < t < \pi$$

together with the usual statement on periodicity

$$f(t + 2\pi) = f(t).$$

Write down an analytic definition for the following periodic function:

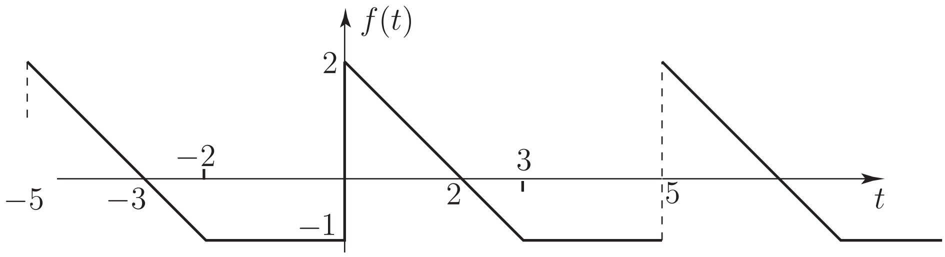

Figure 8

Your solution

We have $f(t) = \left\{ \begin{array}{ll} 2-t & 0<t<3 \\ -1 & 3<t<5 \end{array} \right.$ $\qquad f(t+5) = f(t)$

Sketch the graph of the following periodic functions showing all relevant values:

(i)

$$f(t) = \left\{ \begin{array}{ll} \frac{t^2}{2} & 0<t<4 \\ 8 & 4<t<6 \\ 0 & 6<t<8 \end{array} \right. \qquad f(t+8) = f(t)$$

(ii)

$$f(t) = 2t - t^2 \qquad 0<t<2 \qquad f(t+2) = f(t)$$

Your solution

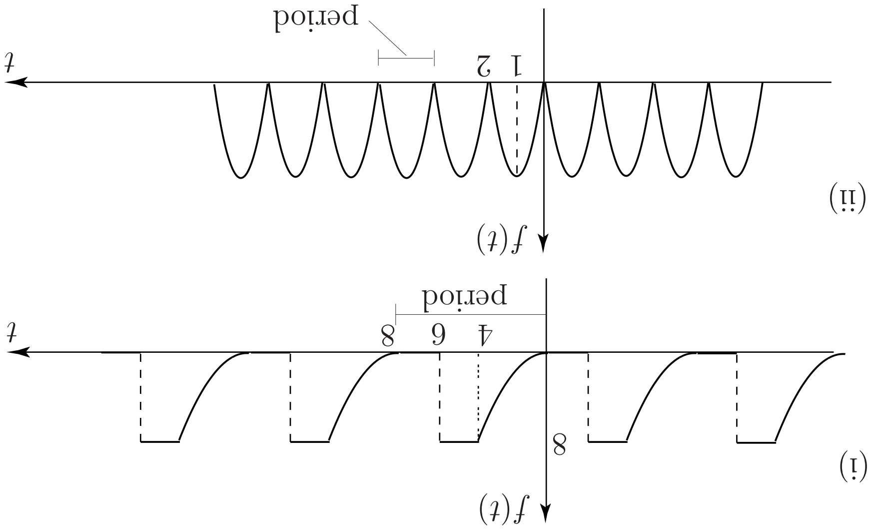

Figure 9