### Even and Odd Functions

#### Introduction

In this Section we examine how to obtain Fourier series of periodic functions which are either even or odd. We show that the Fourier series for such functions is considerably easier to obtain as, if the signal is even only cosines are involved whereas if the signal is odd then only sines are involved. We also show that if a signal reverses after half a period then the Fourier series will only contain odd harmonics.

#### Prerequisites

Before starting this Section you should...

1. know how to obtain a Fourier series
2. be able to integrate functions involving sinusoids
3. have knowledge of integration by parts

#### Learning Outcomes

After completing this Section you should be able to...

- ✓ determine if a function is even or odd or neither
- ✓ easily calculate Fourier coefficients of even or odd functions
1. Even and Odd Functions

We have shown in the previous Section how to calculate, by integration, the coefficients $a_n$ ($n = 0, 1, 2, 3, \ldots$) and $b_n$ ($n = 1, 2, 3, \ldots$) in a Fourier Series. Clearly this is a somewhat tedious process and it is advantageous if we can obtain as much information as possible without recourse to integration.

Referring back to the examples in the previous Section we showed that the square wave (one period of which shown in Figure 1(a)) had a Fourier Series containing a constant term and cosine terms (i.e. all the Fourier coefficients $b_n$ were zero) while the function shown in Figure 1(b) had a more complicated Fourier Series containing both cosine and sine terms as well as a constant.

Contrast the symmetry or otherwise of these two functions.

Your solution

In general a function is called even if its graph is unchanged under reflection in the $y-$axis. This is equivalent to

$$f(-t) = f(t) \quad \text{for all } t$$

Obvious examples of even functions are $t^2, t^4, |t|, \cos t, \cos^2 t, \sin^2 t, \cos nt$.

A function is said to be odd if its graph is symmetrical about the origin. This is equivalent to the condition

$$f(-t) = -f(t)$$
Figure 2 shows an example of an odd function.

![Figure 2](image)

Examples of odd functions are $t$, $t^3$, $\sin t$, $\sin nt$. A periodic function which is odd is the saw-tooth wave in Figure 3.

![Figure 3](image)

Some functions are neither even nor odd. The periodic sawtooth of Figure 1(b) is an example, as is the exponential function $e^t$.

State whether each of the following periodic functions is even or odd or neither.

(i) ![Figure 4](image)

(ii) ![Figure 4](image)

Your solution

(i) is neither even nor odd (with period $2\pi$)

(ii) is odd (with period $\pi$)

(verbosed period) is odd (i))

(verbosed period) is neither even nor odd (i)
A Fourier Series contains a sum of terms while the integral formulae for the Fourier coefficients $a_n$ and $b_n$ contain products of the type $f(t)\cos nt$ and $f(t)\sin nt$ (for the case where $f(t)$ has period $2\pi$.)

We need therefore results for sums and products of functions.

Suppose, for example, $g(t)$ is an odd function and $h(t)$ is an even function.

Let $F_1(t) = g(t)h(t)$ (product of odd and even functions)

so $F_1(-t) = g(-t)h(-t)$ (replacing $t$ by $-t$)

$= (-g(t))h(t)$ (since $g$ is odd and $h$ is even)

$= -g(t)h(t)$

$= -F(t)$

so $F_1(t)$ is odd.

Now suppose $F_2(t) = g(t) + h(t)$ (sum of odd and even functions)

$\therefore F_2(-t) = g(-t) + h(t)$

$= -g(t) + h(t)$

We see that $F_2(-t) \neq F_2(t)$

and $F_2(-t) \neq -F_2(t)$

So $F_2(t)$ is neither even nor odd.

Use similar approaches to the above to investigate sums and products of
(i) two odd functions $g_1(t), g_2(t)$
(ii) two even functions $h_1(t), h_2(t)$

Your solution
If $G_1(t) = g_1(t)g_2(t)$,

$$G_1(-t) = (-g_1(-t))(-g_2(-t)) = g_1(t)g_2(t) = G_1(t),$$

so the product of two odd functions is even.

If $G_2(t) = g_1(t) + g_2(t)$,

$$G_2(-t) = (-g_1(-t)) + (-g_2(-t)) = -g_1(t) - g_2(t) = -G_2(t),$$

so the sum of two odd functions is odd.

If $H_1(t) = h_1(t)h_2(t)$,

$$H_1(-t) = h_1(-t)h_2(-t) = h_1(t)h_2(t) = H_1(t),$$

and

$$H_2(-t) = h_1(-t) + h_2(-t) = -h_1(t) - h_2(t) = -H_2(t),$$

so both the sum and product of two even functions are even.

A similar approach shows that

$$i^\text{even} + (i^\text{even}) = (i^\text{even})$$

$$i^\text{even}i^\text{odd} = (i^\text{even})$$

$$i^\text{odd}i^\text{odd} = (i^\text{even})$$

and

$$i^\text{even} - (i^\text{even}) = (i^\text{even})$$

$$i^\text{even}i^\text{even} = (i^\text{even})$$

$$i^\text{odd}i^\text{even} = (i^\text{even})$$

These results are summarized in the following Key Point.

**Key Point**

**Products of Functions**

- (even) × (even) = (even)
- (even) × (odd) = (odd)
- (odd) × (odd) = (even)

**Sums of Functions**

- (even) + (even) = (even)
- (even) + (odd) = (neither)
- (odd) + (odd) = (odd)
Useful properties of even and of odd functions in connection with integrals can be readily deduced if we recall that a definite integral has the significance of giving us the value of an area:

\[ \int_{a}^{b} f(t) \, dt \]

Figure 5

\[ \int_{a}^{b} f(t) \, dt \] gives us the net value of the shaded area, that above the \( t \)-axis being positive, that below being negative.

For the case of a symmetrical interval \((-a, a)\) deduce what you can about

\[ \int_{-a}^{a} g(t) \, dt \quad \text{and} \quad \int_{-a}^{a} h(t) \, dt \]

where \( g(t) \) is an odd function and \( h(t) \) is an even function.

Figure 6

Your solution

We have

\[ \int_{-b}^{b} f(t) \, dt = \int_{-b}^{b} g(t) \, dt = \int_{-b}^{b} h(t) \, dt \]

(Note that neither result holds for a function which is neither even nor odd.)
2. Fourier Series implications

Since a sum of even functions is itself an even function it is not unreasonable to suggest that a Fourier Series containing only cosine terms (and perhaps a constant term which can also be considered as an even function) can only represent an even periodic function. Similarly a series of sine terms (and no constant) can only represent an odd function. These results can readily be shown more formally using the expressions for the Fourier coefficients $a_n$ and $b_n$.

Recall that for a $2\pi$–periodic function

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt$$

If $f(t)$ is even, deduce whether the integrand is even or odd (or neither) and hence evaluate $b_n$. Repeat for the Fourier coefficients $a_n$.

Your solution

It should be obvious that, for an odd function $f(t)$,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt = \frac{1}{\pi} \int_{-\pi}^{\pi} (\text{odd function}) \, dt = 0$$

$$b_n = \frac{2}{\pi} \int_{0}^{\pi} f(t) \sin nt \, dt$$

Analagous results hold for functions of any period, not necessarily $2\pi$. 

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For a periodic function which is neither even nor odd we can expect at least some of both the $a_n$ and $b_n$ to be non-zero. For example consider the square wave function:

$$f(t) = \begin{cases} 1 & \text{for } 0 \leq t < \pi \\ 0 & \text{for } \pi \leq t < 2\pi \end{cases}$$

Figure 7

This function is neither even nor odd and we have already seen in the previous Section that its Fourier Series contains a constant $\left(\frac{1}{2}\right)$ and sine terms. This result could be expected because we can write

$$f(t) = \frac{1}{2} + g(t)$$

where $g(t)$ is as shown:

Clearly $g(t)$ is odd and will contain only sine terms. The Fourier series are in fact

$$f(t) = \frac{1}{2} + \frac{2}{\pi} \left( \sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t + \ldots \right)$$

and

$$g(t) = \frac{2}{\pi} \left( \sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t + \ldots \right)$$
For each of the following functions deduce whether the corresponding Fourier Series contains

(i) sine or cosine terms or both
(ii) a constant term

Figure 9

Your solution
Confirm the result obtained for part 7 in the last exercise by finding the Fourier Series fully. The function involved is

\[ f(t) = |t| \quad -\pi < t < \pi \]
\[ f(t + 2\pi) = f(t) \]

Your solution

\[
\left( \ldots + \frac{\pi}{2} \cos \frac{\pi}{1} + \frac{\pi}{2} \cos \frac{\pi}{1} + \cos \frac{\pi}{1} \right) \frac{\pi}{\varphi} - \frac{\varphi}{\pi} = (j)f
\]

Also
\[
a_n = \frac{2}{\pi} \int_{-\pi}^{\pi} t \cos nt \, dt
\]
\[
a_0 = \frac{2}{\pi} \int_{-\pi}^{\pi} t \, dt
\]

so the Fourier Series is

\[
\ldots \cos \frac{n\pi}{1} + \cos \frac{3n\pi}{1} + \ldots
\]

After integration by parts

\[
\text{odd } u = \frac{\pi u}{\varphi} - \int \frac{\pi u}{\varphi} \, dt = u_d
\]

\[
\text{even } u = \int \frac{\pi u}{\varphi} \, dt = u_q
\]

Since \( (j)f \) is even we can say immediately

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