Convergence

23.4

Introduction

In this Section we examine, briefly, the convergence characteristics of a Fourier series. We have seen that a Fourier series can be found for functions which are not necessarily continuous (there may be *jumps* in the curve) − the only requirement that we have made is that the function be periodic. We have seen that the more terms we take in the Fourier series the better is the approximation to the given signal. But an obvious question to ask is *what happens at the points of discontinuity?* What does the Fourier series converge to at these points? It must converge to something (finite) since a Fourier series is a sum of very smooth continuous functions. In this Section we give the answer to this question.

Prerequisites

Before starting this Section you should . . .

① know how to obtain a Fourier series

② be familiar with the limit process as applied to functions

Learning Outcomes

After completing this Section you should be able to . . .

✓ determine what a Fourier series converges to at each point, even at a point of discontinuity

✓ use the convergence property of Fourier series to obtain interesting series for the number π

1. Convergence of a Fourier Series

We have now shown how to obtain a Fourier Series for periodic functions. We have suggested that we would expect to be able to approximate such functions by using a "few" terms of the Fourier Series.
The detailed question of the "convergence" or otherwise of Fourier Series has not been discussed. The reason for this is that the great majority of functions likely to be encountered in practice have Fourier Series that do indeed "converge" and can therefore be safely used as approximations.
The precise conditions that have to be fulfilled for a Fourier Series to converge are known as Dirichlet conditions after the French mathematician who investigated the matter. The three conditions are listed in the following Key Point.

Key Point

The Dirichlet conditions for the convergence of a Fourier Series of a periodic function $f(t)$ are:

1. $f(t)$ must have only a finite number of finite discontinuities, within 1 period
2. $f(t)$ must have a finite number of maxima and minima over one period
3. the integral $\displaystyle\int_{-\frac{P}{2}}^{\frac{P}{2}} |f(t)|\mathrm{d}t$ must be finite.

It follows, for example, that if $f(t)$ is defined over $(-\pi, \pi)$ as one of the following functions

$$t^3, \qquad \frac{1}{t-4}, \qquad 3t+2$$

and if $f(t+2\pi) = f(t)$ then $f(t)$ can indeed be represented as a Fourier Series as each function satisfies the Dirichlet conditions.
On the other hand, if, over $(-\pi, \pi)$, $f(t)$ is $\frac{1}{t}$ or $\frac{1}{t-2}$ or $\tan t$ then $f(t)$ cannot be expanded in a Fourier Series because each of these functions has an infinite discontinuity within $(-\pi, \pi)$.
If the Dirichlet conditions are satisfied at a point $t = t_0$ where $f(t)$ is continuous then, as we would expect, the Fourier Series at t_0 viz.

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi t_0}{P}\right) + b_n \sin\left(\frac{n\pi t_0}{P}\right)\right] \qquad \text{converges to the function value } f(t_0)$$

At a point, say $t = t_1$, at which $f(t)$ has a discontinuity then the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi t_1}{P}\right) + b_n \sin\left(\frac{n\pi t_1}{P}\right)\right] \qquad \text{converges to} \quad \frac{1}{2}\{f(t_{1-}) + f(t_{1+})\}$$

where $f(t_{1-})$ is the limit of $f(t)$ as t approaches t_1 from the left and $f(t_{1+})$ is the limit as t approaches t_1 from the right.

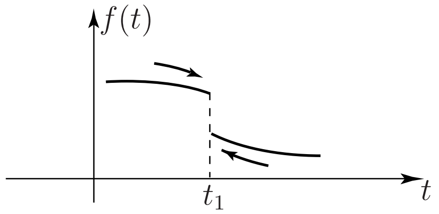

Figure 1

Key Point

If Dirichelet conditions are satisfied then at a point of continuity $t = t_o$

$$f(t_0) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi t_0}{P}\right) + b_n \sin\left(\frac{n\pi t_0}{P}\right)\right]$$

whereas at a point of discontinuity $t = t_1$ the Fourier Series converges to the **average** of the two limiting values

$$\frac{1}{2}\{f(t_{1-}) + f(t_{1+})\} = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi t_1}{P}\right) + b_n \sin\left(\frac{n\pi t_1}{P}\right)\right]$$

Example Suppose we consider the square wave

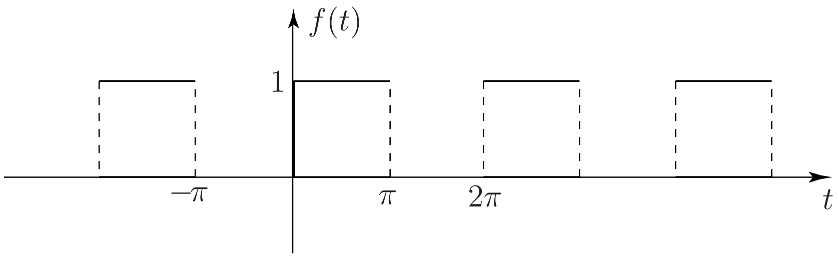

Figure 2

Here $f(t)$ has finite discontinuities at $-\pi, 0$ and π. The Fourier Series of $f(t)$ is (see Section 29.3, p8)

$$\frac{1}{2} + \frac{2}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \ldots\right).$$

At $t = \frac{\pi}{2}$, for example, where the function $f(t)$ is continuous the square wave converges to $f\left(\frac{\pi}{2}\right) = 1$. On the other hand at $t = \pi$ the Series clearly has the value $\frac{1}{2}$ (since all the sine terms are zero here). This value $\frac{1}{2}$ agrees with the average of the 2 limiting values of $f(t)$ at $t = \frac{\pi}{2}$ viz. $\frac{1}{2}(1+0) = \frac{1}{2}$. If we actually put $t = \frac{\pi}{2}$ in the Fourier Series we obtain

$$\begin{aligned} & \frac{1}{2} + \frac{2}{\pi}\left(\sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi}{2}\right) + \ldots\right) \\ = & \frac{1}{2} + \frac{2}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right) \end{aligned}$$

Since the series converges, as we have seen, to $f\left(\frac{\pi}{2}\right) = 1$, we obtain the useful result

$$\frac{1}{2} + \frac{2}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right) = 1$$

$$\text{or} \qquad 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$$

The function

$$f(t) = \begin{cases} 0 & -\pi < t < 0 \\ t^2 & 0 < t < \pi \end{cases}$$

$$f(t + 2\pi) = f(t)$$

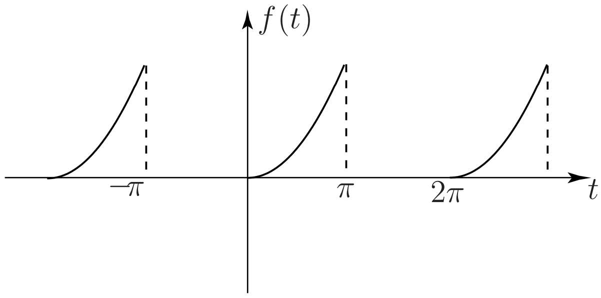

Figure 3

has Fourier Series (see Section 29.2, p21)

$$\begin{aligned} \frac{\pi^2}{6} & - 2\left(\cos t - \frac{\cos 2t}{4} + \frac{\cos 3t}{9} - \ldots\right) \\ & + \left\{\left(\pi - \frac{4}{\pi}\right)\sin t - \frac{\pi}{2}\sin 2t + \left(\frac{\pi}{3} - \frac{4}{9\pi}\right)\sin 3t - \frac{\pi}{4}\sin 4t + \ldots\right\} \end{aligned}$$

By using a suitable value of t show that

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots = \frac{\pi^2}{6}$$

First decide on the appropriate value of t to use:

Your solution

Looking at the Fourier Series the series we seek is present in the cosine terms so we wish to remove the sine terms. This we can do by selecting $t = 0$ or $t = \pi$. The choice $t = 0$ will make the cosine terms become:

$$\left(1 - \frac{1}{4} + \frac{1}{9} - \cdots\right)$$

which is not what we seek. Hence we put $t = \pi$.

Now put $t = \pi$ in the series and decide what the Fourier Series will converge to at this value. Hence complete the question.

Your solution

At $t = \pi$ the Fourier Series is

$$\frac{\pi^2}{6} - 2\left(\cos\pi - \frac{\cos 2\pi}{4} + \frac{\cos 3\pi}{9} - \cdots\right)$$

$$= \frac{\pi^2}{6} - 2\left(-1 - \frac{1}{4} - \frac{1}{9} - \frac{1}{16} - \cdots\right) = \frac{\pi^2}{6} + 2\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right)$$

At $t = \pi$ the Fourier Series will converge to

$$\frac{1}{2}\left(\pi^2 + 0\right) = \frac{\pi^2}{2}$$

(the average of the left and right hand limits)

Hence

$$\frac{\pi^2}{6} + 2\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right) = \frac{\pi^2}{2}$$

$$\therefore \qquad 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{1}{2}\left(\frac{\pi^2}{2} - \frac{\pi^2}{6}\right) = \frac{\pi^2}{6}$$

as required.

Note that if we do use $t = 0$ in the Fourier Series (which converges to $f(0) = 0$) we obtain another infinite series but with alternating signs:

$$\frac{\pi^2}{6} - 2\left(1 - \frac{1}{4} + \frac{1}{9} - \ldots\right) = 0$$

or

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots \qquad = \frac{\pi^2}{12}$$

Exercises

1. Obtain the Fourier series of

   $$f(t) = |t| \qquad -\pi \leq t \leq \pi$$

   $$f(t + 2\pi) = f(t)$$

   By putting $t = 0$ show that

   $$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

2. (a) Obtain the Fourier series of the 2π periodic function

   $$f(t) = \frac{t^2}{4} \qquad -\pi \leq t \leq \pi$$

   and use it to obtain the following identities:

   $$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

   $$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

   (b) Show that $\quad 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} \cdots = \frac{\pi^2}{8}$

3. Obtain the Fourier series of the 2π periodic function

   $$f(t) = t \qquad -\pi \leq t \leq \pi$$

   Use the series to show that

   $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

1. $\dfrac{\pi}{2} + \sum_{n=1}^{\infty} \dfrac{(-4)}{(2n-1)^2\pi} \cos[(2n-1)t]$

2. (a) $\dfrac{\pi^2}{12} + \sum_{n=1}^{\infty} \dfrac{\cos(n\pi)}{n^2} \cos nt$

   (b) add the two series from (a).

3. $-2 \sum_{n=1}^{\infty} \dfrac{(-1)^n}{n} \sin nt$