

# The Complex Form

23.6



## Introduction

In this Section we show how a Fourier series can be expressed more concisely if we introduce the complex number  $i = \sqrt{-1}$ . By utilising the Euler formula:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

we can replace the trigonometric functions by complex exponential functions. By also combining the Fourier coefficients  $a_n$  and  $b_n$  into a complex coefficient  $c_n$  through

$$c_n = \frac{1}{2}(a_n - ib_n)$$

we find that, for a given periodic signal, both sets of constants can be found in one operation. We also obtain Parseval's Theorem which has important applications in electrical engineering. The complex formulation of a Fourier series is an important pre-cursor of the Fourier transform which attempts to Fourier analyse non-periodic functions.



## Prerequisites

Before starting this Section you should ...

- ① know how to obtain a Fourier series
- ② be familiar with the complex numbers
- ③ be familiar with the relation between the exponential function and the trigonometric functions



## Learning Outcomes

After completing this Section you should be able to ...

- ✓ express a periodic function in terms of its Fourier series in complex form
- ✓ understand Parseval's theorem

# 1. Complex Exponential Form of a Fourier Series

So far we have discussed the **trigonometric** form of a Fourier Series i.e. we have represented functions of period  $T$  in the terms of sinusoids, and possibly a constant term, using

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \left( \frac{2n\pi t}{T} \right) + b_n \sin \left( \frac{2n\pi t}{T} \right) \right).$$

If we use the angular frequency

$$\omega_0 = \frac{2\pi}{T}$$

We obtain the more concise form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t).$$

We have seen that the Fourier coefficients are calculated using the following integrals:

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos n\omega_0 t \, dt \quad n = 0, 1, 2, \dots \quad (1)$$

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin n\omega_0 t \, dt \quad n = 1, 2, \dots \quad (2)$$

An alternative, more concise form, of a Fourier Series is available using complex quantities. This form is quite widely used by Engineers, for example in Circuit Theory and Control Theory, and, in this course, will lead naturally into a later Section on the Fourier Transform.

# 2. Revision of the exponential form of a complex number

Recall that a complex number in Cartesian form

$$z = a + ib,$$

where  $a$  and  $b$  are real numbers and  $i^2 = -1$ , can be written in **polar** form

$$z = r(\cos \theta + i \sin \theta)$$

where  $r = |z| = \sqrt{a^2 + b^2}$  and  $\theta$ , the argument or phase of  $z$ , is such that

$$a = r \cos \theta \quad b = r \sin \theta.$$

A more concise version of the polar form of  $z$  can be obtained by defining a **complex exponential** quantity  $e^{i\theta}$  by

$$e^{i\theta} = \cos \theta + i \sin \theta$$

(This is sometimes known as Euler's relation.) The polar angle  $\theta$  is normally expressed in **radians**.

Replacing  $i$  by  $-i$  we obtain

$$e^{-i\theta} = \cos \theta - i \sin \theta$$



Write down in cosine, sine form (i)  $e^{i\pi/6}$  (ii)  $e^{-i\pi/6}$ . Then write each in Cartesian form.

**Your solution**

$$\frac{z}{1} - \frac{z}{\sqrt{3}}i = \left(\frac{9}{\pi}\right) \sin i - \left(\frac{9}{\pi}\right) \cos = e^{-i\pi/6} \quad \frac{z}{1} + \frac{z}{\sqrt{3}}i = \left(\frac{9}{\pi}\right) \sin i + \left(\frac{9}{\pi}\right) \cos = e^{i\pi/6}$$

We have, by definition,



Write down (i)  $\cos\left(\frac{\pi}{6}\right)$  (ii)  $\sin\left(\frac{\pi}{6}\right)$  in terms of  $e^{i\pi/6}$  and  $e^{-i\pi/6}$ .

**Your solution**

$$\left(\frac{9}{\pi}\right) \sin i = \frac{z}{1} (e^{i\pi/6} - e^{-i\pi/6}) \quad \text{or} \quad \left(\frac{9}{\pi}\right) \sin i = \frac{z}{1} (e^{i\pi/6} - e^{-i\pi/6})$$

Similarly, subtracting the two results,

$$\left(\frac{9}{\pi}\right) \cos = \frac{z}{1} (e^{i\pi/6} + e^{-i\pi/6}) \quad \text{or} \quad \left(\frac{9}{\pi}\right) \cos = \frac{z}{1} (e^{i\pi/6} + e^{-i\pi/6})$$

We have, adding the two results from the previous guided exercise

(Don't forget the factor  $i$  in this latter case)

Clearly similar calculations could be carried out for any angle  $\theta$  and the results are summarised in the following Key Point.



### Key Point

$$\cos \theta = \frac{1}{2} (e^{i\theta} + e^{-i\theta}) \quad \sin \theta = \frac{1}{2i} (e^{i\theta} - e^{-i\theta})$$

Using these results we can redraft an expression of the form

$$a_n \cos n\theta + b_n \sin n\theta$$

in terms of complex exponentials.

(This expression, with  $\theta = \omega_0 t$ , is of course the  $n$ 'th harmonic of a trigonometric Fourier Series.)



Using the results from the last Key Point (with  $n\theta$  instead of  $\theta$ ) rewrite

$$a_n \cos n\theta + b_n \sin n\theta$$

in complex exponential form.

First do direct substitution

**Your solution**

$$(e^{in\theta} - e^{-in\theta}) \frac{1}{2i} + (e^{in\theta} + e^{-in\theta}) \frac{1}{2} = \cos n\theta + i \sin n\theta$$

or

$$(e^{in\theta} - e^{-in\theta}) \frac{1}{2i} = \sin n\theta \quad (e^{in\theta} + e^{-in\theta}) \frac{1}{2} = \cos n\theta$$

We have

Now collect the terms in  $e^{in\theta}$  and in  $e^{-in\theta}$

**Your solution**

$$\begin{aligned} & \theta u i^{-1} e^{(u q i + u v)} \frac{z}{1} + \theta u i e^{(u q i - u v)} \frac{z}{1} \\ & \theta u i^{-1} e^{\left(\frac{1}{u q} - u v\right)} \frac{z}{1} + \theta u i e^{\left(\frac{1}{u q} + u v\right)} \frac{z}{1} \end{aligned}$$

or, since  $i^{-1} = \frac{1}{i} = \frac{1}{1} \cdot \frac{1}{i}$

We get

We can now write this expression in more concise form by defining

$$c_n = \frac{1}{2}(a_n - i b_n)$$

which has complex conjugate

$$c_n^* = \frac{1}{2}(a_n + i b_n).$$

Hence, we have finally,

**Your solution**

$$\theta u i^{-1} e^{u q} + \theta u i e^{u q} = \theta u \cos u q + \theta u \sin u q$$

Clearly, we can now rewrite the trigonometric Fourier Series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

as

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (c_n e^{in\omega_0 t} + c_n^* e^{-in\omega_0 t}) \quad (3)$$

A neater, and particularly concise, form of this expression can be obtained as follows: Firstly write  $\frac{a_0}{2} = c_0$  (which is consistent with the general definition of  $c_n$  since  $b_0 = 0$ ) The second term in the summation

$$\sum_{n=1}^{\infty} c_n^* e^{-in\omega_0 t} = c_1^* e^{-i\omega_0 t} + c_2^* e^{-2i\omega_0 t} + \dots$$

can be written, if we define  $c_{-n} = c_n^* = \frac{1}{2}(a_n + i b_n)$ , as

$$c_{-1} e^{-i\omega_0 t} + c_{-2} e^{-2i\omega_0 t} + c_{-3} e^{-3i\omega_0 t} + \dots = \sum_{n=-1}^{-\infty} c_n e^{in\omega_0 t}$$

Hence (3) can be written

$$c_0 + \sum_{n=1}^{\infty} c_n e^{in\omega_0 t} + \sum_{n=-1}^{-\infty} c_n e^{in\omega_0 t}$$

or, finally, in the extremely concise form

$$\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}.$$

The **complex Fourier coefficients**  $c_n$  can be readily obtained as follows using (1) and (2) for  $a_n, b_n$ .

Firstly

$$c_0 = \frac{a_0}{2} = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) dt \quad (4)$$

For  $n = 1, 2, 3, \dots$  we have

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)(\cos n\omega_0 t - i \sin n\omega_0 t) dt$$

$$\text{i.e.} \quad c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-in\omega_0 t} dt \quad (5)$$

Also for  $n = 1, 2, 3, \dots$  we have

$$c_{-n} = c_n^* = \frac{1}{2}(a_n + ib_n) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{in\omega_0 t} dt$$

This last expression is equivalent to stating that for  $n = -1, -2, -3, \dots$

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-in\omega_0 t} dt \quad (6)$$

The 3 equations (4), (5), (6) can thus all be contained in the one expression

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-in\omega_0 t} dt \quad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \dots$$

The results of this discussion are summarised in the following Key Point.



### Key Point

#### Fourier Series in Complex Form

A function  $f(t)$  of period  $T$  has a complex Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t} \quad \text{where} \quad c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-in\omega_0 t} dt$$

For the special case  $T = 2\pi$  so that  $\omega_0 = 1$  these formulae become particularly simple

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int} \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt.$$

Note the extremely concise form of these results.

### 3. Properties of the complex Fourier coefficients

Using properties of the trigonometric Fourier coefficients  $a_n, b_n$  we can readily deduce the following results for the  $c_n$ 's:

1.  $c_0 = \frac{a_0}{2}$  is always real
2. Suppose the periodic function  $f(t)$  is even so that all  $b_n$  are zero. Then since, in the complex form, the  $b_n$  arise as the imaginary part of  $c_n$ , it follows that for  $f(t)$  even the coefficients  $c_n$  ( $n = \pm 1, \pm 2, \dots$ ) are wholly real.



If  $f(t)$  is odd, what can you deduce about the Fourier coefficients  $c_n$ ?

**Your solution**

Since, for an odd periodic function the Fourier coefficients  $a_n$  (which constitute the real part of  $c_n$ ) are zero, then in this case the complex coefficients  $c_n$  are wholly imaginary.

### Further important points about complex Fourier coefficients

1. Since

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-in\omega_0 t} dt$$

then if  $f(t)$  is even, so  $c_n$  will be real, we have two possible methods for evaluating  $c_n$ :

- (a) evaluate the integral above **as it stands** i.e. over the full range  $(-\frac{T}{2}, \frac{T}{2})$ : note carefully that the second term in the integrand is neither an even nor an odd function so the integrand itself is

(even function)  $\times$  (neither even nor odd function) = neither even nor odd function.

Thus we **cannot** write

$$c_n = \frac{2}{T} \int_0^{T/2} f(t) e^{-in\omega_0 t} dt$$

(b) Put  $e^{-in\omega_0 t} = \cos n\omega_0 t - i \sin n\omega_0 t$  so

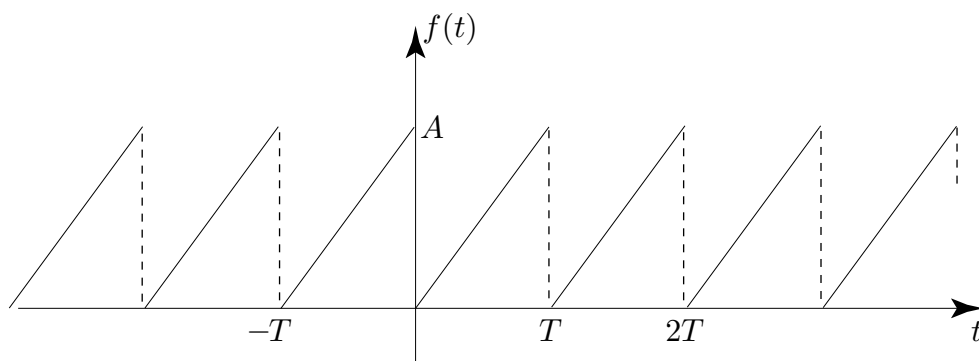
$$\begin{aligned} f(t)e^{-in\omega_0 t} &= f(t) \cos n\omega_0 t - i f(t) \sin n\omega_0 t \\ &= (\text{even})(\text{even}) - i(\text{even})(\text{odd}) \\ &= (\text{even}) - i(\text{odd}). \end{aligned}$$

Hence

$$c_n = \frac{2}{T} \int_0^{\frac{T}{2}} f(t) \cos n\omega_0 t \, dt = \frac{a_n}{2}.$$

2. If  $f(t + \frac{T}{2}) = -f(t)$  then of course only **odd** harmonic coefficients  $c_n$  ( $n = \pm 1, \pm 3, \pm 5, \dots$ ) will arise in the complex Fourier Series just as with trigonometric series.

**Example** Find the complex Fourier Series of the sawtooth wave shown:



### Solution

We have

$$\begin{aligned} f(t) &= \frac{At}{T} \quad 0 < t < T \\ f(t + T) &= f(t) \end{aligned}$$

The period is  $T$  in this case so

$$\omega_0 = \frac{2\pi}{T}.$$

Looking at the graph of  $f(t)$  we can say immediately

- i. the Fourier Series will contain a constant term  $c_0$
- ii. if we imagine shifting the horizontal axis up to  $\frac{A}{2}$  the signal can be written

$$f(t) = \frac{A}{2} + g(t),$$

where  $g(t)$  is an odd function with complex Fourier coefficients that are purely imaginary.



### Solution (contd.)

Hence we expect the required complex Fourier series of  $f(t)$  to contain a constant term  $\frac{A}{2}$  and complex exponential terms with purely imaginary coefficients. We have, from the general theory, and using  $0 < t < T$  as the basic period for integrating.

$$c_n = \frac{1}{T} \int_0^T \frac{At}{T} e^{-in\omega_0 t} dt = \frac{A}{T^2} \int_0^T t e^{-in\omega_0 t} dt$$

We can evaluate the integral using parts:

$$\begin{aligned} \int_0^T t e^{-in\omega_0 t} dt &= \left[ \frac{t e^{-in\omega_0 t}}{(-in\omega_0)} \right]_0^T + \frac{1}{in\omega_0} \int_0^T e^{-in\omega_0 t} dt \\ &= \frac{T e^{in\omega_0 T}}{(-in\omega_0)} - \frac{1}{(in\omega_0)^2} [e^{-in\omega_0 t}]_0^T \end{aligned}$$

But  $\omega_0 = \frac{2\pi}{T}$  so

$$\begin{aligned} e^{-in\omega_0 T} = e^{-in2\pi} &= \cos 2n\pi - i \sin 2n\pi \\ &= 1 - i0 = 1 \end{aligned}$$

Hence the integral becomes

$$\frac{T}{-in\omega_0} - \frac{1}{(in\omega_0)^2} \{e^{-in\omega_0 T} - 1\}$$

Hence

$$c_n = \frac{A}{T^2} \left( \frac{T}{-in\omega_0} \right) = \frac{iA}{2\pi n} \quad n = \pm 1, \pm 2, \dots$$

Note that

$$c_{-n} = \frac{iA}{2\pi(-n)} = \frac{-iA}{2\pi n} = c_n^* \quad \text{as it must}$$

Also  $c_0 = \frac{1}{T} \int_0^T \frac{At}{T} dt = \frac{A}{2}$  as expected.

Hence the required complex Fourier Series is

$$f(t) = \frac{A}{2} + \frac{iA}{2\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{in\omega_0 t}}{n}$$

which could be written, showing only the constant and the first two harmonics as

$$f(t) = \frac{A}{2\pi} \left\{ \dots - i \frac{e^{-i2\omega_0 t}}{2} - i e^{-i\omega_0 t} + \pi + i e^{i\omega_0 t} + i \frac{e^{i2\omega_0 t}}{2} + \dots \right\}.$$

The corresponding trigonometric Fourier Series for the function can be readily obtained from this complex series by combining the terms in  $\pm n$ ,  $n = 1, 2, 3, \dots$

For example this first harmonic is

$$\begin{aligned} \frac{A}{2\pi} \{-ie^{-i\omega_0 t} + ie^{i\omega_0 t}\} \\ &= \frac{A}{2\pi} \{-i(\cos \omega_0 t - i \sin \omega_0 t) + i(\cos \omega_0 t + i \sin \omega_0 t)\} \\ &= \frac{A}{2\pi} (-2 \sin \omega_0 t) = -\frac{A}{\pi} \sin \omega_0 t \end{aligned}$$

Performing similar calculations on the other harmonics we obtain the trigonometric form of the Fourier series

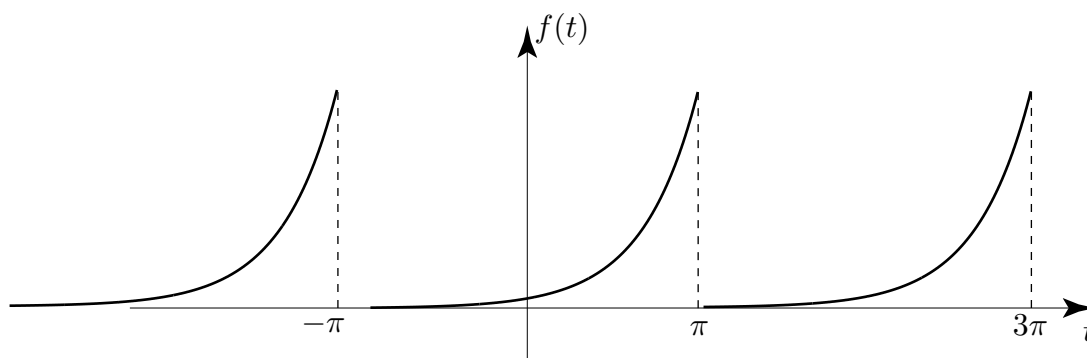
$$f(t) = \frac{A}{2} - \frac{A}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\omega_0 t}{n}.$$



Find the complex Fourier Series of the  $2\pi$ -periodic function

$$f(t) = e^t \quad -\pi < t < \pi$$

$$f(t + 2\pi) = f(t)$$



Firstly write down an integral expression for the Fourier coefficients  $c_n$ :

**Your solution**

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^t e^{-int} dt$$

We have, since  $T = 2\pi$  so  $\omega_0 = 1$

Now combine the real exponential and the complex exponential as one term and carry out the integration

**Your solution**

$$c_n = \int_{-\pi}^{\pi} \frac{2\pi}{1} e^{i(u-1)t} e^{-\pi(u-1)t} \frac{2\pi}{1} e^{i(u-1)t} dt = \int_{-\pi}^{\pi} \frac{2\pi}{1} e^{i(u-1)t} e^{-\pi(u-1)t} dt$$

We have

Now simplify this as far as possible and write out the Fourier Series:

**Your solution**

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{\pi}{\sinh \pi} (1+n)^n (1+n_2)^n e^{i n t} \quad \text{since } \cos n\pi = (-1)^n.$$

This includes the constant term  $c_0 = \frac{\pi}{\sinh \pi}$ . Hence the required Fourier Series is

$$c_{-n} = \frac{\pi}{\sinh \pi} \frac{(1-n)^n (1+n_2)^n}{\cos(-n\pi)} = \frac{\pi}{\sinh \pi} \frac{(1-n)^n (1+n_2)^n}{\cos n\pi} = c_n^* \text{ as required.}$$

Also

Note that the coefficients  $c_n$   $n = \pm 1, \pm 2, \dots$  have both real and imaginary parts in this case as the function being expanded is neither even nor odd.

$$c_n = \frac{1}{1} \frac{2\pi}{1} (1-in)^n (e^{\pi} - e^{-\pi}) \cos n\pi = \frac{\pi}{\sinh \pi} \frac{(1-in)^n (1+n_2)^n}{\cos n\pi}$$

Hence

$$e^{i(1-in)\pi} = e^{\pi} e^{-in\pi} = e^{\pi} (\cos n\pi - i \sin n\pi) = e^{\pi} \cos n\pi$$

$$e^{-i(1-in)\pi} = e^{-\pi} e^{in\pi} = e^{-\pi} (\cos n\pi + i \sin n\pi) = e^{-\pi} \cos n\pi$$

## 4. Parseval's Theorem

This is essentially a mathematical theorem but has, as we shall see, an important engineering interpretation particularly in electrical engineering.

Parseval's Theorem states that if  $f(t)$  is a periodic function with period  $T$  and if  $c_n$  ( $n = 0, \pm 1, \pm 2, \dots$ ) denote the complex Fourier coefficients of  $f(t)$  then

$$\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t) dt = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

In words the theorem states that the mean square value of the signal  $f(t)$  over one period equals the sum of the squared magnitudes of all the complex Fourier coefficients.

### Proof of Parseval's Theorem.

Assume  $f(t)$  has a complex Fourier Series of the usual form:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t} \quad \left( \omega_0 = \frac{2\pi}{T} \right)$$

where

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-in\omega_0 t} dt$$

Then

$$f^2(t) = f(t)f(t) = f(t) \sum c_n e^{in\omega_0 t} = \sum c_n f(t) e^{in\omega_0 t} dt$$

Hence

$$\begin{aligned} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t) dt &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum c_n f(t) e^{in\omega_0 t} dt \\ &= \frac{1}{T} \sum c_n \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{in\omega_0 t} dt \\ &= \sum c_n c_n^* \\ &= \sum_{n=-\infty}^{\infty} |c_n|^2 \end{aligned}$$

which completes the proof.

Parseval's Theorem can also be written in terms of the Fourier coefficients  $a_n, b_n$  of the trigonometric Fourier Series.

Recall that

$$c_0 = \frac{a_0}{2} \quad c_n = \frac{a_n - ib_n}{2} \quad n = 1, 2, 3, \dots \quad c_n = \frac{a_n + ib_n}{2} \quad n = -1, -2, -3, \dots$$

so

$$|c_n|^2 = \frac{a_n^2 + b_n^2}{4} \quad n = \pm 1, \pm 2, \pm 3, \dots$$

so

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{a_0^2}{4} + 2 \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{4}$$

and hence Parseval's Theorem becomes

$$\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t) dt = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \quad \dots (*)$$

The engineering interpretation of this theorem is as follows:

suppose  $f(t)$  denotes an electrical signal (current or voltage): then from elementary circuit theory  $f^2(t)$  is the instantaneous power (in a 1 ohm resistor) so that

$$\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t) dt$$

is the energy dissipated in the resistor during one period.

Now a sinusoid wave of the form

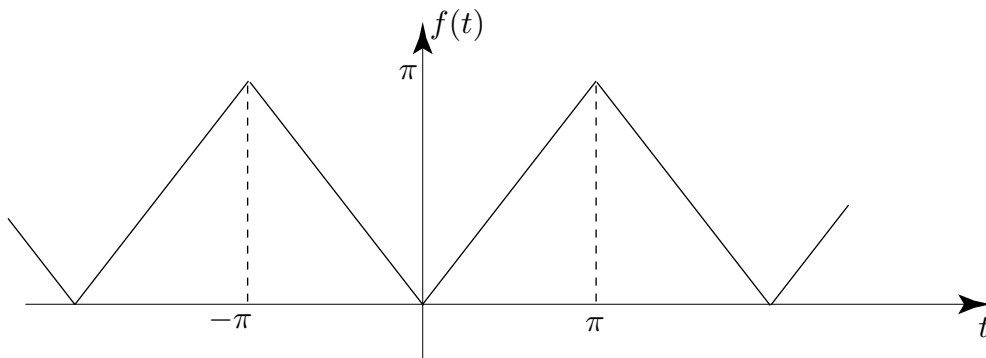
$$A \cos \omega t \quad (\text{or } A \sin \omega t)$$

has a mean square value  $\frac{A^2}{2}$  so a purely sinusoidal signal would dissipate a power  $\frac{A^2}{2}$  in a 1 ohm resistor.

Hence Parseval's Theorem in the form (\*) states that the average power dissipated over 1 period equals the sum of the powers of the constant (or d.c.) components and of all the sinusoidal (or alternating) components.



The triangular signal shown



has trigonometric Fourier Series (see Section 29.3, p10)

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{\substack{n=1 \\ (\text{odd } n)}}^{\infty} \frac{\cos nt}{n^2}.$$

Use Parseval's Theorem to show that  $\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{96}$ .

Firstly, identify  $a_0$ ,  $a_n$  and  $b_n$  for this situation and write down the definition of  $f(t)$  for this case.

**Your solution**

Also

$$f(t) = (t + 2\pi) f(t)$$

$$|t| = t - \pi > t > \pi$$

$$\left. \begin{aligned} 0 &= b_n & n = 1, 2, 3, 4, \dots \\ 0 &= a_n & n = 2, 4, 6, \dots \\ -\frac{n\pi}{4} &= a_n & n = 1, 3, 5, \dots \end{aligned} \right\}$$

We have  $\frac{2}{\pi} = \frac{2}{\pi}$

Now evaluate the integral on the left hand side of Parseval's Theorem and hence complete the problem.

**Your solution**

Hence

$$\frac{3}{\pi^2} = \frac{4}{\pi^2} + \frac{1}{8} \sum_{n=1}^{\infty} \frac{1}{n^4} \quad \dots \quad \frac{12}{\pi^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^4} \quad \dots \quad \frac{96}{\pi^4} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^4}$$

The right hand side of Parseval's Theorem is

$$\frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_2(t) dt = \int_{-\pi}^{\pi} t^2 dt = \frac{1}{3} \left[ \frac{t^3}{3} \right]_{-\pi}^{\pi} = \frac{2}{3} \pi^3$$

We have  $f_2(t) = t^2$  so

$$\frac{4}{\pi^2} + \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2}{3} \pi^3$$

### Exercises

Obtain the complex Fourier series for each of the following functions of period  $2\pi$

1.  $f(t) = t \quad -\pi \leq t \leq \pi$
2.  $f(t) = t \quad 0 \leq t \leq 2\pi$
3.  $f(t) = e^t \quad -\pi \leq t \leq \pi$

1.  $\sum_{n=1}^{\infty} \frac{1}{n} e^{-in} \quad (\text{sum from } -\infty \text{ to } \infty \text{ excluding } n = 0).$
2.  $\sum_{n=1}^{\infty} \frac{1}{n} e^{in} \quad (\text{sum from } -\infty \text{ to } \infty \text{ excluding } n = 0).$
3.  $\sum_{n=1}^{\infty} \frac{1}{n} e^{-in} \quad (\text{sum from } -\infty \text{ to } \infty).$