Application of Fourier Series

Introduction

In this Section we look at a typical application of Fourier series. The problem we study is that of a differential equation with a periodic (but non-sinusoidal) forcing function. The differential equation chosen models, in particular, a lightly damped vibrating system.

Prerequisites

Before starting this Section you should...

1. know how to obtain a Fourier series
2. be familiar with the complex numbers
3. be familiar with the relation between the exponential function and the trigonometric functions

Learning Outcomes

After completing this Section you should be able to...

✓ solve a linear differential equation with a periodic forcing function via Fourier series.
Vibration problems are often modelled by ordinary differential equations with constant coefficients. For example, the motion of a spring with stiffness $k$ and damping constant $c$ is modelled by

$$m \frac{d^2 y}{dt^2} + c \frac{dy}{dt} + ky = 0$$

(1)

where $y(t)$ is the displacement of a mass $m$ connected to the spring. It is well-known that if $c^2 < 4mk$, usually referred to as the lightly damped case, then

$$y(t) = e^{-\alpha t} (A \cos \omega t + B \sin \omega t)$$

(2)

i.e. the motion is sinusoidal but damped by the negative exponential term. In (2) we have used the notation

$$\alpha = \frac{c}{2m}, \quad \omega = \frac{1}{2m} \sqrt{4km - c^2}$$

The values of $A$ and $B$ depend upon initial conditions.

The system represented by (1), and whose solution in (2), is referred to as an unforced damped harmonic oscillator.

A lightly damped oscillator driven by a time-dependent forcing function $F(t)$ is modelled by the differential equation

$$m \frac{d^2 y}{dt^2} + c \frac{dy}{dt} + ky = F(t)$$

(3)

The solution or system response in (3) has two parts:

(a) A transient solution of the form (2),

(b) A forced or steady state solution whose form, of course, depends on $F(t)$.

If $F(t)$ is sinusoidal:

$$F(t) = A \sin(\Omega t + \phi) \quad \text{where} \quad \Omega \text{ and } \phi \text{ are constants.}$$

then the steady state solution is fairly readily obtained by standard techniques for solving differential equations. If $F(t)$ is periodic but non-sinusoidal then Fourier series may be used to obtain the steady state solution. The method is based on the principle of superposition which is actually applicable to any linear (homogeneous) differential equation. (Another engineering application is the series $LCR$ circuit with an applied periodic voltage.)

The principle of superposition is easily demonstrated:-

Let $y_1(t)$ and $y_2(t)$ be the steady state solutions of (3) when $F(t) = F_1(t)$ and $F(t) = F_2(t)$ respectively. Then

$$m \frac{d^2 y_1}{dt^2} + c \frac{dy_1}{dt} + ky_1 = F_1(t)$$

$$m \frac{d^2 y_2}{dt^2} + c \frac{dy_2}{dt} + ky_2 = F_2(t)$$

Simply adding these equations we obtain

$$m \frac{d^2}{dt^2} (y_1 + y_2) + c \frac{d}{dt} (y_1 + y_2) + k(y_1 + y_2) = F_1(t) + F_2(t)$$

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from which it follows that if \( F(t) = F_1(t) + F_2(t) \) then the system response is the sum \( y_1(t) + y_2(t) \). This, in its simplest form, is the principle of superposition. More generally if the forcing function is

\[
F(t) = \sum_{n=1}^{N} F_n(t)
\]

then the response is \( y(t) = \sum_{n=1}^{N} y_n(t) \) where \( y_n(t) \) is the response to the forcing function \( F_n(t) \).

Returning to the specific case where \( F(t) \) is periodic, the solution procedure for the steady state response is as follows:

(a) Obtain the Fourier series of \( F(t) \).

(b) Solve (3) for the response \( y_n(t) \) corresponding to the \( n \)th harmonic in the Fourier series. (The response \( y_0 \) to the constant term, if any, in the Fourier series may have to be obtained separately).

(c) **Superpose** the solutions obtained to give the overall steady-state motion:

\[
y(t) = y_0(t) + \sum_{n=1}^{N} y_n(t)
\]

The procedure can be lengthy but the solution is of great engineering interest because if the frequency of one harmonic in the Fourier series is close to the *natural frequency* \( \sqrt{\frac{k}{m}} \) of the undamped system then the response to that harmonic will dominate the solution.

This exercise is quite long but it will provide useful practice in applying Fourier series to a practical problem. Essentially you should follow steps (a), (b), (c) above carefully.

The problem is to find the steady-state response \( y(t) \) of a spring/mass/damper system modelled by

\[
m \frac{d^2y}{dt^2} + c \frac{dy}{dt} + ky = F(t)
\]

where \( F(t) \) is the **periodic square wave** function shown in Figure 9.

Step 1: Obtain the Fourier series of \( F(t) \) noting that it is an odd function.
Your solution

\begin{align*}
\text{(where the sum is over odd } n \text{ only)} \quad & \frac{u}{\sin \pi n} \sum_{n=1}^{\infty} \frac{\nu}{\nu F} = (i) J \\
\text{even } u & \begin{cases} 0 \\ \frac{x u}{\nu F} \end{cases} = \\
\text{odd } u & \frac{\nu u}{\nu F} = \\
(\nu u \cos - 1) & \frac{\nu u}{\nu F} = \\
0 & \frac{\nu u}{\nu F} \sin - 1 = \\
0 & \frac{\nu u}{\nu F} \\
\cdots & \text{odd } n = 1
\end{align*}

The calculation is similar to those you have performed earlier in this workbook.

**Step 2(a)**
Since each term in the Fourier series is a sine term you must now solve (4) to find the steady-state response \( y_n \) to the \( n^{th} \) harmonic input:

\[ F_n(t) = b_n \sin n \omega t \quad n = 1, 3, 5, \ldots \]
From the basic theory of linear differential equations this response has the form

\[ y_n = A_n \cos n\omega t + B_n \sin n\omega t \] (5)

where \( A_n \) and \( B_n \) are coefficients to be determined by substituting (5) into (4) with \( F(t) = F_n(t) \). Do this to obtain simultaneous equations for \( A_n \) and \( B_n \).

**Your solution**

\[ uq = uG(\sigma u w) + uV(\sigma u) \]
\[ 0 = uG(\sigma u) + uV(\sigma u) \]

Then, by comparing coefficients of \( \cos n\omega t \) and \( \sin n\omega t \), we obtain the pair of simultaneous equations:

\[ (\sigma u)n = \mu n \sin(\mu u) + \nu \mu u \cos(\mu u) \]
\[ 0 = \mu \mu u + \nu \mu u \cos(\mu u) \]

From which, substituting into (4) and collecting terms in \( \cos n\omega t \) and \( \sin n\omega t \),

\[ (\sigma u)n = \mu n \sin(\mu u) - \nu (\mu u) \cos(\mu u) \]
\[ 0 = \mu \mu u + \nu \mu u \cos(\mu u) \]

We have, differentiating (5),

\[ \frac{\partial}{\partial t} y_n = \frac{\partial}{\partial t} \left( A_n \cos n\omega t + B_n \sin n\omega t \right) \]

\[ \frac{\partial^2}{\partial t^2} y_n = \frac{\partial^2}{\partial t^2} \left( A_n \cos n\omega t + B_n \sin n\omega t \right) \]

**Step 2(b)**

Now solve (6) and (7) to obtain \( A_n \) and \( B_n \).

**Your solution**
\[ A_n = -c_\omega n b_n (k - m_\omega^2 n_2) + \omega_2 n c_2 \] (8)

\[ B_n = (k - m_\omega^2 n_2) b_n (k - m_\omega^2 n_2) + \omega_2 n c_2 \] (9)

where we have written \( \omega_n \) for \( n\omega \) as the frequency of the \( n \)th harmonic.

It follows that the steady-state response \( y_n \) to the \( n \)th harmonic of the Fourier series of the forcing function is given by (5). The amplitudes \( A_n \) and \( B_n \) are given by (8) and (9) respectively in terms of the systems parameters \( k, c, m \), the frequency \( \omega_n \) of the harmonic and its amplitude \( b_n \).

In practice it is more convenient to represent \( y_n \) in the so-called amplitude/phase form:

\[ y_n = C_n \sin(\omega_n t + \phi_n) \] (10)

where, from (5) and (10),

\[ A_n \cos \omega_n t + B_n \sin \omega_n t = C_n (\cos \phi_n \sin \omega_n t + \sin \phi_n \cos \omega_n t). \]

Hence

\[ C_n \sin \phi_n = A_n \quad C_n \cos \phi_n = B_n \]

so

\[ \tan \phi_n = \frac{A_n}{B_n} = \frac{c_\omega n}{(m_\omega^2 n - k)^2} \] (11)

\[ C_n = \sqrt{A_n^2 + B_n^2} = \frac{b_n}{\sqrt{(m_\omega^2 n - k)^2 + \omega_2 n c^2}} \] (12)

Step 3

Finally, by the superposition principle, the complete steady state response of the system to the forcing function of Figure 9, is

\[ y(t) = \sum_{n=1}^{\infty} y_n(t) = \sum_{n=1}^{\infty} (n \text{ odd}) C_n (\sin \omega_n t + \phi_n) \]

where \( C_n \) and \( \phi_n \) are given by (11) and (12).

In practice, since \( b_n = \frac{4F_0}{n\pi} \) it follows that the amplitude \( C_n \) also decreases as \( \frac{1}{n} \). However, if one of the harmonic frequencies say \( \omega'_n \) is close to the natural frequency \( \sqrt{\frac{k}{m}} \) of the undamped oscillator then that particular frequency harmonic will dominate in the steady-state response. The particular value \( \omega'_n \) will of course depend on the values of the system parameters \( k \) and \( m \).