Contents 24

Fourier **transforms**

1. The Fourier transform
2. Properties of the Fourier Transform
3. Some Special Fourier Transform Pairs

Learning **outcomes**

needs doing

Time **allocation**

You are expected to spend approximately thirteen hours of independent study on the material presented in this workbook. However, depending upon your ability to concentrate and on your previous experience with certain mathematical topics this time may vary considerably.

The Fourier Transform 24.1

Introduction

Prerequisites

Before starting this Section you should . . .

①

Learning Outcomes

After completing this Section you should be able to . . .

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1. The Fourier Transform

The Fourier Transform is a mathematical technique that has extensive applications in Science and Engineering, for example in Physical Optics, Chemistry (e.g. Nuclear Magnetic Resonance), Communications Theory and Linear Systems Theory.
Unlike Fourier series which, as we have seen in the previous two units, is mainly useful for periodic functions, the Fourier Transform (FT for short) permits alternative representations of, mostly, non-periodic functions.
We shall firstly derive the Fourier Transform from the complex exponential form of the Fourier Series and then study various properties of the FT.

2. Informal Derivation of the Fourier Transform

Recall that if $f(t)$ is a period T function, which we will temporarily re-write as $f_T(t)$ for emphasis, then we can expand it in a complex Fourier Series,

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{\mathrm{i}n\omega_0 t} \qquad (1)$$

where $\omega_0 = \frac{2\pi}{T}$. In words, harmonics of frequency $n\omega_0 = n\frac{2\pi}{T} \quad n = 0, \pm 1, \pm 2, \ldots$ are present in the series and these frequencies are separated by

$$n\omega_0 - (n-1)\omega_0 = \omega_0 = \frac{2\pi}{T}.$$

Hence, as T increases the frequency separation becomes smaller and can be conveniently written as $\Delta\omega$. This suggests that as $T \to \infty$, corresponding to a non-periodic function then $\Delta\omega \to 0$ and the frequency representation contains **all** frequency harmonics.

To see this in a little more detail, we recall (Workbook 23: Fourier Series) that the complex Fourier coefficients c_n are given by

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t) e^{-\mathrm{i}n\omega_0 t}\mathrm{d}t. \qquad (2)$$

Putting $\frac{1}{T}$ as $\frac{\omega_0}{2\pi}$ and then substituting (2) in (1) we get

$$f_T(t) = \sum_{n=-\infty}^{\infty}\left\{\frac{\omega_0}{2\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t)e^{-\mathrm{i}n\omega_0 t}\mathrm{d}t\right\} e^{\mathrm{i}n\omega_0 t}.$$

In view of the discussion above, as $T \to \infty$, we can put ω_0 as $\Delta\omega$ and replace the sum over the discrete frequencies $n\omega_0$ by an integral over all frequencies. We replace $n\omega_0$ by a general frequency variable ω. We then obtain the double integral representation

$$f(t) = \int_{-\infty}^{\infty}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-\mathrm{i}\omega t}\mathrm{d}t\right\} e^{\mathrm{i}\omega t}\mathrm{d}\omega. \qquad (3)$$

The inner integral (over all t) will give a function dependent only on ω which we write as $F(\omega)$. Then (3) can be written

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{\mathrm{i}\omega t}\mathrm{d}\omega. \qquad (4)$$

where

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-\mathrm{i}\omega t}\mathrm{d}t. \tag{5}$$

The representation (4) of $f(t)$ which involves all frequencies ω can be considered as the equivalent for a non-periodic function of the complex Fourier Series representation (1) of a periodic function. The expression (5) for $F(\omega)$ is analogous to the relation (2) for the Fourier coefficients c_n.
The function $F(\omega)$ is called the **Fourier Transform** of the function $f(t)$. Symbolically we can write

$$F(\omega) = \mathcal{F}\{f(t)\}.$$

Equation (4) enables us, in principle, to write $f(t)$ in terms of $F(\omega)$. $f(t)$ is often called the **inverse Fourier Transform** of $F(\omega)$ and we can denote this by writing

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\}.$$

Looking at the basic relation (3) it is clear that the position of the factor $\frac{1}{2\pi}$ is somewhat arbitrary in (4) and (5). If instead of (5) we define

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-\mathrm{i}\omega t}\mathrm{d}t.$$

then (4) must be written

$$f(t) = \int_{-\infty}^{\infty} F(\omega)e^{\mathrm{i}\omega t}\mathrm{d}\omega.$$

A third, and more symmetric, alternative is to write

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-\mathrm{i}\omega t}\mathrm{d}t$$

and, consequently,

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{\mathrm{i}\omega t}\mathrm{d}\omega.$$

We shall use (4) and (5) throughout this section but you should be aware of these other possibilities which might be used in other texts.
Engineers often refer to $F(\omega)$ (whichever precise definition is used!) as the **frequency domain** representation of a function or signal and $f(t)$ as the **time domain** representation. In what follows we shall use this language where appropriate. However, (5) is really a mathematical transformation for obtaining one function from another and (4) is then the inverse transformation for recovering the initial function. In some applications of Fourier Transforms (which we shall not study) the time/frequency interpretations are not relevant. However, in engineering applications, such as communications theory, the frequency representation is often used very literally.
As can be seen above, notationally we will use capital letters to denote Fourier Transforms: thus a function $f(t)$ has a Fourier transform denoted by $F(\omega)$, $g(t)$ a Fourier transform written $G(\omega)$ and so on. The notation $F(\mathrm{i}\omega)$, $G(\mathrm{i}\omega)$ is used in some texts because ω occurs in (5) only in the term $e^{-\mathrm{i}\omega t}$.

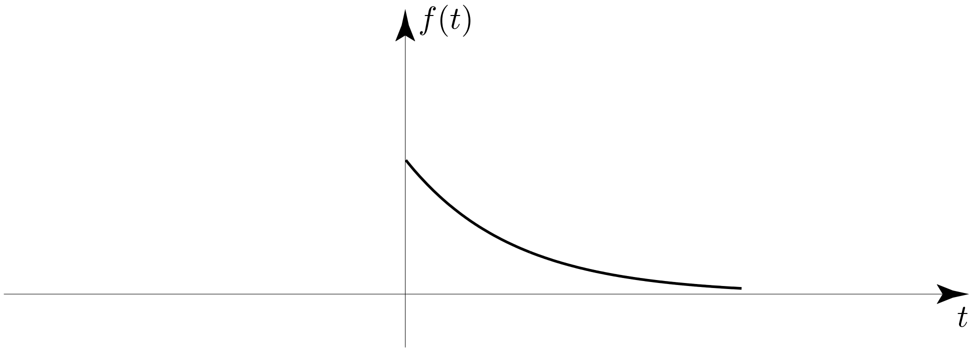
3. Existence of the Fourier Transform

We will discuss this question in a little detail at a later stage when we will also take up briefly the relation between the Fourier Transform and the Laplace Transform (Workbook 20) which you have met earlier.
For now we will use (5) to obtain the Fourier Transforms of some important functions.

Example Find the Fourier Transform of the one-sided exponential function

$$f(t) = \begin{cases} 0 & t < 0 \\ e^{-\alpha t} & t > 0 \end{cases}$$

where α is a positive constant.



Note that if $u(t)$ is used to denote the Heaviside unit step function viz.

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

then we can write

$$f(t) = e^{-\alpha t}u(t).$$

(We shall frequently use this concise notation for one-sided functions.)

Solution

Using (5) then by straightforward integration

$$\begin{aligned} F(\omega) &= \int_0^\infty e^{-\alpha t}e^{-\mathrm{i}\omega t}\mathrm{d}t \qquad \text{(since } f(t) = 0 \text{ for } t < 0) \\ &= \int_0^\infty e^{-(\alpha+\mathrm{i}\omega t)}\mathrm{d}t \\ &= \left[\frac{e^{-(\alpha+\mathrm{i}\omega)t}}{-(\alpha+\mathrm{i}\omega)}\right]_0^\infty \\ &= \frac{1}{\alpha+\mathrm{i}\omega} \end{aligned}$$

since $e^{-\alpha t} \to 0$ as $t \to \infty$ for $\alpha > 0$.

This important Fourier Transform is written in the Key Point

Key Point

$$\mathcal{F}\{e^{-\alpha t}u(t)\} = \frac{1}{\alpha + \mathrm{i}\omega}, \qquad \alpha > 0.$$

Note that this **real** function has a **complex** Fourier Transform.

Write down the Fourier Transforms of
(i) $e^{-t}u(t)$ (ii) $e^{-3t}u(t)$ (iii) $e^{-\frac{t}{2}}u(t)$

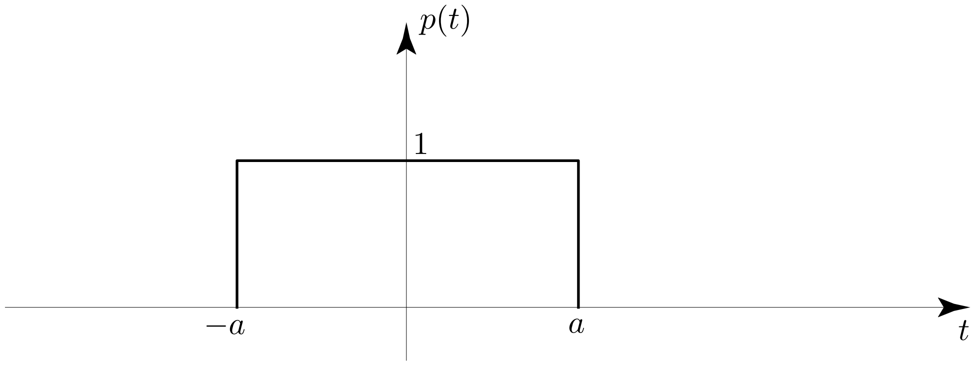
Your solution

We have using the general result just proved:
(i) $\alpha = 1$ so $\mathcal{F}\{e^{-t}u(t)\} = \frac{1}{1+\mathrm{i}\omega}$
(ii) $\alpha = 3$ so $\mathcal{F}\{e^{-3t}u(t)\} = \frac{1}{3+\mathrm{i}\omega}$
(iii) $\alpha = \frac{1}{2}$ so $\mathcal{F}\{e^{-\frac{t}{2}}u(t)\} = \frac{1}{\frac{1}{2}+\mathrm{i}\omega}$

Obtain, using the integral definition (5), the Fourier Transform of the rectangular pulse

$$p(t) = \begin{cases} 1 & -a < t < a \\ 0 & \text{otherwise} \end{cases} .$$

Note that the pulse width is $2a$.



First write down using (5) the integral from which the transform will be calculated.

Your solution

We have $P(\omega) \equiv \mathcal{F}\{p(t)\} = \displaystyle\int_{-a}^{a} (1)e^{-\mathrm{i}\omega t}dt$ using the definition of $p(t)$

Now evaluate this integral and write down the final Fourier Transform in trigonometric, rather than complex exponential form.

Your solution

We have

$$P(\omega) = \int_{-a}^{a} (1)e^{-\mathrm{i}\omega t}dt = \left[\frac{e^{-\mathrm{i}\omega t}}{(-\mathrm{i}\omega)}\right]_{-a}^{a} = \frac{e^{-\mathrm{i}\omega a} - e^{+\mathrm{i}\omega a}}{(-\mathrm{i}\omega)}$$

$$= \frac{(\cos\omega a - \mathrm{i}\sin\omega a) - (\cos\omega a + \mathrm{i}\sin\omega a)}{(-\mathrm{i}\omega)} = \frac{2\mathrm{i}\sin\omega a}{\mathrm{i}\omega}$$

i.e.

$$P(\omega) = \mathcal{F}\{p(t)\} = \frac{2\sin\omega a}{\omega} \qquad (6)$$

Note that in this case the Fourier Transform is **wholly real**.

Engineers often call the function $\frac{\sin x}{x}$ the sinc function. Consequently if we write, the transform (6) of the rectangular pulse as

$$P(\omega) = 2a\frac{\sin\omega a}{\omega a},$$

we can say

$$P(\omega) = 2a\mathrm{sinc}(\omega a).$$

Using the result (6) in (4) we have the **Fourier Integral representation** of the rectangular pulse.

$$p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\frac{\sin\omega a}{\omega}e^{\mathrm{i}\omega t}d\omega.$$

As we have already mentioned, this corresponds to a Fourier series representation for a periodic function.

Key Point

The Fourier Transform of a Rectangular Pulse

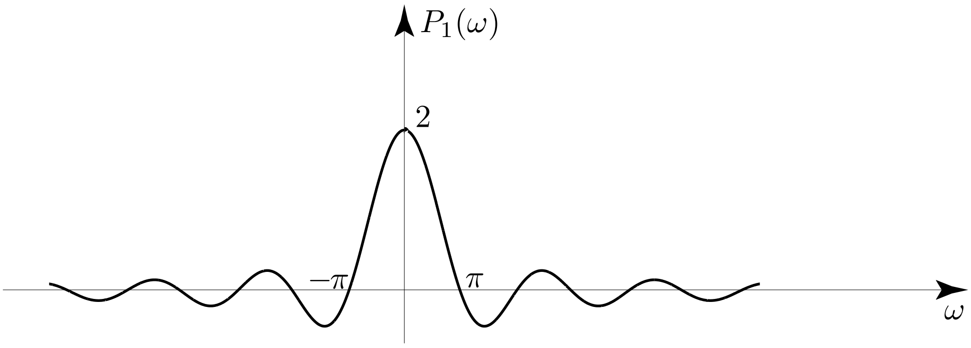
If $p_a(t) = \begin{cases} 1 & -a < t < a \\ 0 & \text{otherwise} \end{cases}$ then:

$$\mathcal{F}\{p_a(t)\} = 2a\frac{\sin \omega a}{\omega a} = 2a\text{sinc}(\omega a)$$

Clearly, if the rectangular pulse has width 2, corresponding to $a = 1$ we have:

$$P_1(\omega) \equiv \mathcal{F}\{p_1(t)\} = 2\frac{\sin \omega}{\omega}.$$

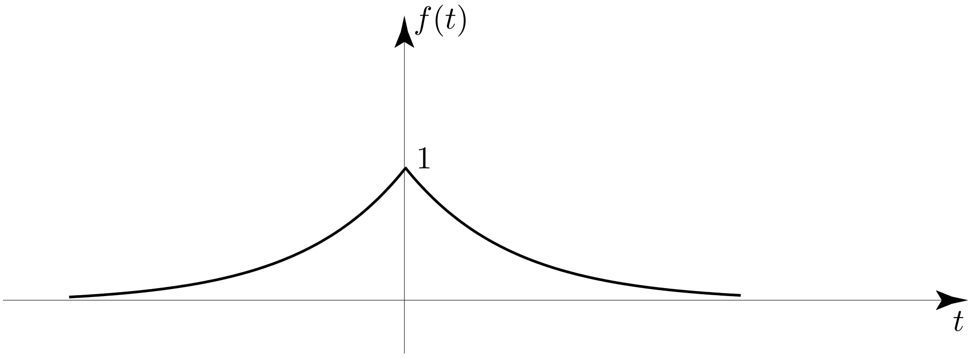
As $\omega \to 0$, then $2\frac{\sin\omega}{\omega} \to 2$. Also, the function $2\frac{\sin\omega}{\omega}$ is an even function being the product of two odd functions $2\sin\omega$ and $\frac{1}{\omega}$. The graph of $P_1(\omega)$ is as follows:



Obtain the Fourier Transform of the **two sided** exponential function

$$f(t) = \begin{cases} e^{\alpha t} & t < 0 \\ e^{-\alpha t} & t > 0 \end{cases}$$

where α is a positive constant.



Your solution

We must separate the range of the integrand into $[-\infty, 0]$ and $[0, \infty]$ since the function $f(t)$ is defined separately in these two regions: then

$$F(\omega) = \int_{-\infty}^{0} e^{\alpha t}e^{-\mathrm{i}\omega t}\mathrm{d}t + \int_{0}^{\infty} e^{-\alpha t}e^{-\mathrm{i}\omega t}\mathrm{d}t = \int_{-\infty}^{0} e^{(\alpha-\mathrm{i}\omega)t}\mathrm{d}t + \int_{0}^{\infty} e^{-(\alpha+\mathrm{i}\omega)t}\mathrm{d}t$$

$$= \left[\frac{e^{(\alpha-\mathrm{i}\omega)t}}{(\alpha-\mathrm{i}\omega)}\right]_{-\infty}^{0} + \left[\frac{e^{-(\alpha+\mathrm{i}\omega)t}}{-(\alpha+\mathrm{i}\omega)}\right]_{0}^{\infty}$$

$$= \frac{1}{\alpha-\mathrm{i}\omega} + \frac{1}{\alpha+\mathrm{i}\omega} = \frac{2\alpha}{\alpha^2+\omega^2}.$$

Note that, as in the case of the rectangular pulse, we have here a real even function of t giving a Fourier Transform which is wholly real. Also, in both cases, the Fourier Transform is an **even** (as well as real) function of ω.
Note also that it follows from the above calculation that

$$\mathcal{F}\{e^{-\alpha t}u(t)\} = \frac{1}{\alpha + \mathrm{i}\omega}$$

(as we have already found) and

$$\mathcal{F}\{e^{\alpha t}u(-t)\} = \frac{1}{\alpha - \mathrm{i}\omega}$$

where

$$e^{\alpha t}u(-t) = \begin{cases} e^{\alpha t} & t < 0 \\ 0 & t > 0 \end{cases}.$$

4. Properties of the Fourier Transform

1. Real and Imaginary Parts of a Fourier Transform

Using the definition (5) we have,

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-\mathrm{i}\omega t}\mathrm{d}t.$$

If we write $e^{-\mathrm{i}\omega t} = \cos\omega t - \mathrm{i}\sin\omega t$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t\ \mathrm{d}t - \mathrm{i}\int_{-\infty}^{\infty} f(t)\sin\omega t\ \mathrm{d}t$$

where both integrals are real, assuming that $f(t)$ is real.
Hence the real and imaginary parts of the Fourier transform are:

$$\text{Re }(F(\omega)) = \int_{-\infty}^{\infty} f(t)\cos\omega t \mathrm{d}t \qquad \text{Im }(F(\omega)) = -\int_{-\infty}^{\infty} f(t)\sin\omega t \mathrm{d}t.$$

Recalling that if $h(t)$ is even and $g(t)$ is odd then $\int_{-a}^{a} h(t)\mathrm{d}t = 2\int_{0}^{a} h(t)\mathrm{d}t$ and $\int_{-a}^{a} g(t)\mathrm{d}t = 0$, deduce $\text{Re}(F(\omega))$ and $\text{Im}(F(\omega))$ if

(i) $f(t)$ is a real even function

(ii) $f(t)$ is a real odd function

Your solution

for (i)

If $f(t)$ is real and even

$$R(\omega) \equiv \text{Re } F(\omega) = 2\int_{0}^{\infty} f(t)\cos\omega t \mathrm{d}t$$

(because the integrand is even)

$$I(\omega) \equiv \text{Im } F(\omega) = -\int_{-\infty}^{\infty} f(t)\sin\omega t \mathrm{d}t = 0$$

(because the integrand is odd).

Thus, any real even function $f(t)$ has a wholly real Fourier Transform. Also since

$$\cos((-\omega)t) = \cos(-\omega t) = \cos\omega t$$

the Fourier Transform in this case will be a real even function.

Your solution

for (ii)

Now

$$\text{Re } F(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t\,dt$$

$$= \int_{-\infty}^{\infty} (\text{odd})(\text{even})dt = \int_{-\infty}^{\infty} (\text{odd})dt = 0$$

and

$$\text{Im } F(\omega) = -\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt$$

$$= -2\int_{0}^{\infty} f(t)\sin\omega t\,dt$$

(because the integrand is (odd)(odd)=(even)).

Also since $\sin((-\omega)t) = -\sin\omega t$, the Fourier Transform in this case is an odd function of ω.

These results are summarised in the following Key Point:

Key Point

$f(t)$	$F(\omega) = \mathcal{F}\{f(t)\}$
real and even	real and even
real and odd	purely imaginary and odd
neither even nor odd	complex, $F(\omega) = R(\omega) + \text{i}I(\omega)$

2. Polar Form of a Fourier Transform

Example We have shown that the one-sided exponential function,

$$f(t) = e^{-\alpha t}u(t)$$

has Fourier Transform

$$F(\omega) = \frac{1}{\alpha + \mathrm{i}\omega}.$$

Find the real and imaginary parts of $F(\omega)$ for this case.

Your solution

We have

$$F(\omega) = \frac{1}{\alpha + \mathrm{i}\omega} = \frac{\alpha - \mathrm{i}\omega}{\alpha^2 + \omega^2}$$

(after rationalising i.e. multiplying numerator and denominator by $\alpha - \mathrm{i}\omega$)

Hence

$$R(\omega) = \text{Re } F(\omega) = \frac{\alpha}{\alpha^2 + \omega^2} \quad \text{(even function of } \omega\text{)}$$

$$I(\omega) = \text{Im } F(\omega) = \frac{-\omega}{\alpha^2 + \omega^2} \quad \text{(odd function of } \omega\text{)}$$

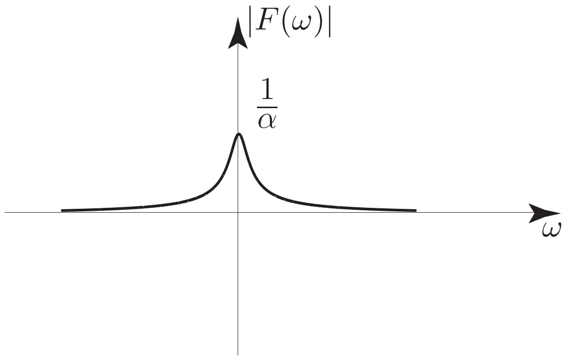
We can rewrite $F(\omega)$, like any other complex quantity, in **polar** form by calculating the magnitude and the argument (or phase):

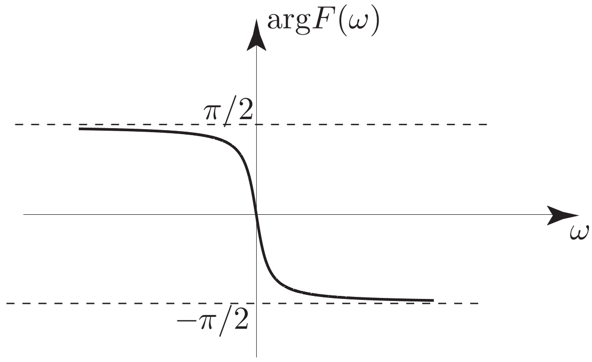
$$|F(\omega)| = \sqrt{R^2(\omega) + I^2(\omega)}$$

$$= \sqrt{\frac{\alpha^2 + \omega^2}{(\alpha^2 + \omega^2)^2}} = \frac{1}{\sqrt{\alpha^2 + \omega^2}}$$

and

$$\arg F(\omega) = \tan^{-1}\frac{I(\omega)}{R(\omega)} = \tan^{-1}\left(\frac{-\omega}{\alpha}\right).$$





In general, a Fourier Transform whose Cartesian form is

$$F(\omega) = R(\omega) + \mathrm{i}I(\omega)$$

has a polar form

$$F(\omega) = |F(\omega)|e^{\mathrm{i}\phi(\omega)}$$

where $\phi(\omega) \equiv \arg\, F(\omega)$.
Graphs, such as those shown, of $|F(\omega)|$ and $\arg\, F(\omega)$ plotted against ω are often referred to as **magnitude** and **phase** spectra, respectively.