Properties of the Fourier Transform

24.2

Introduction

Prerequisites

Before starting this Section you should . . .

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Learning Outcomes

After completing this Section you should be able to . . .

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1. Linearity Properties of the Fourier Transform

(i) If $f(t)$, $g(t)$ are functions with transforms $F(\omega),\ G(\omega)$, respectively, then

- $\mathcal{F}\{f(t) + g(t)\} = F(\omega) + G(\omega)$

i.e. if we add 2 functions then the Fourier Transform of the resulting function is simply the sum of the individual Fourier Transforms.

(ii) If k is any constant,

- $\mathcal{F}\{kf(t)\} = kF(\omega)$

i.e. if we multiply a function by any constant then we must multiply the Fourier Transform by the same constant. These properties follow from the definition of the Fourier Transform and properties of integrals.

Examples

1.
$$\begin{aligned}
\mathcal{F}\{2e^{-t}u(t) + 3e^{-2t}u(t)\} &= \mathcal{F}\{2e^{-t}u(t)\} + \mathcal{F}\{3e^{-2t}u(t)\} \\
&= 2\mathcal{F}\{e^{-t}u(t)\} + 3\mathcal{F}\{e^{-2t}u(t)\} \\
&= \frac{2}{1+\mathrm{i}\omega} + \frac{3}{2+\mathrm{i}\omega}
\end{aligned}$$

2.
$$\begin{aligned}
\text{If} \qquad f(t) &= \begin{cases} 4 & -3 \leq t \leq 3 \\ 0 & \text{otherwise} \end{cases} \\
\text{then} \qquad f(t) &= 4p_3(t) \\
\text{and so} \qquad F(\omega) &= 4P_3(\omega) = \frac{8}{\omega}\sin 3\omega
\end{aligned}$$

using the standard result for $\mathcal{F}\{p_a(t)\}$.

If $f(t) = \begin{cases} 6 & -2 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$ write down $F(\omega)$.

Your solution

We have $f(t) = 6p_2(t)$ so $F(\omega) = \frac{12}{\omega}\sin 2\omega$.

2. Shift properties of the Fourier Transform

There are two basic shift properties of the Fourier Transform:
(i) Time shift property:

$$\mathcal{F}\{f(t-t_0)\} = e^{-\mathrm{i}\omega t_0}F(\omega)$$

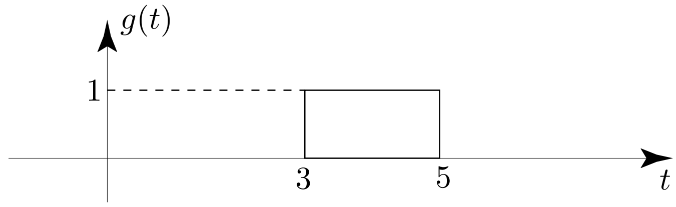
(ii) Frequency shift property

$$\mathcal{F}\{e^{\mathrm{i}\omega_0 t}f(t)\} = F(\omega-\omega_0).$$

Here t_0, ω_0 are constants. In words, shifting (or translating) a function in one domain corresponds to a multiplication by a complex exponential function in the other domain.
We omit the proofs of these properties which follow from the definition of the Fourier Transform.

Example Use the time-shifting property to find the Fourier Transform of the function

$$g(t) = \begin{cases} 1 & 3 \le t \le 5 \\ 0 & \text{otherwise} \end{cases}$$



Solution

$g(t)$ is a pulse of width 2 and can be obtained by shifting the symmetrical rectangular pulse

$$p_1(t) = \begin{cases} 1 & -1 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

by 4 units to the right.
Hence by putting $t_0 = 4$ in the time shift theorem

$$G(\omega) = \mathcal{F}\{g(t)\} = e^{-4\mathrm{i}\omega}\frac{2}{\omega}\sin\omega.$$

Verify the above result by direct integration.

Your solution

We have

$$\begin{aligned} G(\omega) &= \int_3^5 1 e^{-i\omega t} dt \\ &= \left[\frac{e^{-i\omega t}}{-i\omega}\right]_3^5 = \frac{e^{-5i\omega} - e^{-3i\omega}}{-i\omega} = e^{-4i\omega}\left(\frac{e^{-i\omega} - e^{i\omega}}{-i\omega}\right) \\ &= e^{-4i\omega}\left(\frac{e^{i\omega} - e^{-i\omega}}{i\omega}\right) = e^{-4i\omega} 2\frac{\sin\omega}{\omega}, \end{aligned}$$

as obtained using the time-shift property.

Use the frequency shift property to obtain the Fourier Transform of the **modulated wave**

$$g(t) = f(t)\cos\omega_0 t$$

where $f(t)$ is an arbitrary signal whose Fourier Transform is $F(\omega)$.

First rewrite $g(t)$ in terms of complex exponentials.

Your solution

We have

$$g(t) = f(t)\left(\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}\right) = \frac{1}{2}f(t)e^{i\omega_0 t} + \frac{1}{2}f(t)e^{-i\omega_0 t}$$

Now use the linearity property and the frequency shift property on each term to obtain $G(\omega)$.

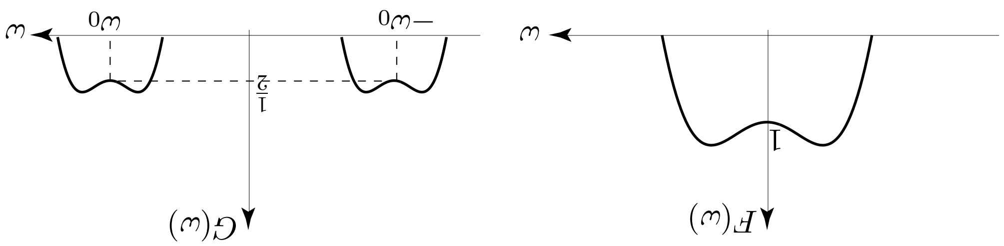
Your solution

We have, by linearity

$$\mathcal{F}\{g(t)\} = \frac{1}{2}\mathcal{F}\{f(t)e^{\mathrm{i}\omega_0 t}\} + \frac{1}{2}\mathcal{F}\{f(t)e^{-\mathrm{i}\omega_0 t}\}$$

and by the frequency shift property

$$G(\omega) = \frac{1}{2}F(\omega - \omega_0) + \frac{1}{2}F(\omega + \omega_0).$$



3. Inversion of the Fourier Transform

Formal inversion of the Fourier Transform, i.e. finding $f(t)$ for a given $F(\omega)$ is sometimes possible using the inversion integral (4). However, in elementary cases, we can use a Table of standard Fourier Transforms together, if necessary, with the appropriate properties of the Fourier Transform.

Example Find the inverse Fourier Transform of $F(\omega) = 20\dfrac{\sin 5\omega}{5\omega}$.

Solution

The appearance of the sine function implies that $f(t)$ is a symmetric rectangular pulse.
We know the standard form

$$\mathcal{F}\{p_a(t)\} = 2a\frac{\sin \omega a}{\omega a}$$

or

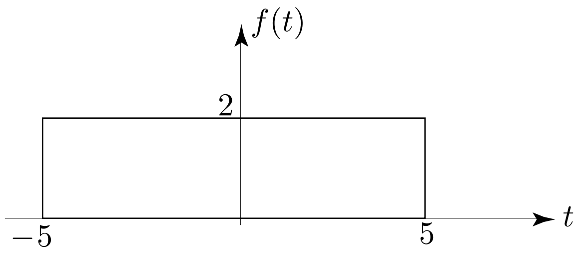
$$\mathcal{F}^{-1}\{2a\frac{\sin \omega a}{\omega a}\} = p_a(t).$$

Putting $a = 5$

$$\mathcal{F}^{-1}\{10\frac{\sin 5\omega}{5\omega}\} = p_5(t).$$

Thus, by the linearity property

$$f(t) = \mathcal{F}^{-1}\{20\frac{\sin 5\omega}{5\omega}\} = 2p_5(t)$$



Example Find the inverse Fourier Transform of $G(\omega) = 20\frac{\sin 5\omega}{5\omega}\exp(-3i\omega)$.

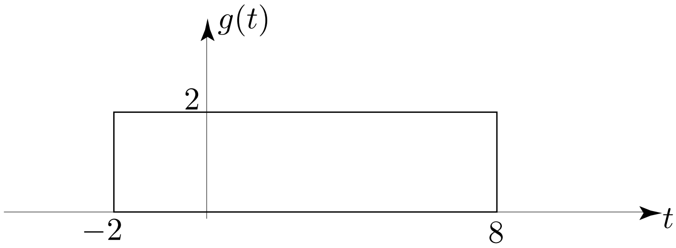
Solution

The occurrence of the complex exponential factor in the FT suggests the time-shift property with the time shift $t_0 = +3$ (i.e. a right shift).
From the previous example

$$\mathcal{F}^{-1}\{20\frac{\sin 5\omega}{5\omega}\} = 2p_5(t)$$

so

$$g(t) = \mathcal{F}^{-1}\{20\frac{\sin 5\omega}{5\omega}e^{-3i\omega}\} = 2p_5(t-3)$$



Find the inverse Fourier Transform of

$$H(\omega) = 6\frac{\sin 2\omega}{\omega}e^{-4\mathrm{i}\omega}.$$

Firstly ignore the exponential factor and "de-Fourier" (to coin a phrase) the remaining terms:

Your solution

We have

$$\mathcal{F}^{-1}\{2a\frac{\sin \omega a}{\omega a}\} = p_a(t)$$

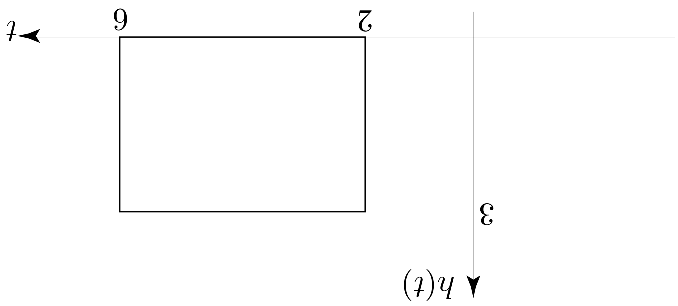
so putting $a = 2$ $\quad \mathcal{F}^{-1}\{2\frac{\sin 2\omega}{\omega}\} = p_2(t) \quad \therefore \quad \mathcal{F}^{-1}\{6\frac{\sin 2\omega}{\omega}\} = 3p_2(t)$

Now take account of the exponential factor:

Your solution

Using the time-shift theorem for $t_0 = 4$

$$h(t) = \mathcal{F}^{-1}\{6\frac{\sin 2\omega}{\omega}e^{-4\mathrm{i}\omega}\} = 3p_2(t-4)$$



Example Find the inverse Fourier Transform of

$$K(\omega) = \frac{2}{1 + 2(\omega - 1)\mathrm{i}}$$

Solution

The presence of the term $(\omega - 1)$ instead of ω suggests the frequency shift property.
Hence, we consider first

$$\hat{K}(\omega) = \frac{2}{1 + 2\mathrm{i}\omega}.$$

The relevant standard form is

$$\mathcal{F}\{e^{-\alpha t}u(t)\} = \frac{1}{\alpha + \mathrm{i}\omega}$$

or

$$\mathcal{F}^{-1}\{\frac{1}{\alpha + \mathrm{i}\omega}\} = e^{-\alpha t}u(t).$$

Hence, writing $\hat{K}(\omega) = \dfrac{1}{\frac{1}{2} + \mathrm{i}\omega}$

$$\hat{k}(t) = e^{-\frac{1}{2}t}u(t).$$

Then, by the frequency shift property with $\omega_0 = 1$

$$k(t) = \mathcal{F}^{-1}\{\frac{2}{1 + 2(\omega - 1)\mathrm{i}}\} = e^{-\frac{1}{2}t}e^{\mathrm{i}t}u(t).$$

Here $k(t)$ is a complex time-domain signal.

Find the inverse Fourier Transforms of

(i) $L(\omega) = 2\dfrac{\sin\{3(\omega - 2\pi)\}}{(\omega - 2\pi)}$ (2) $M(\omega) = \dfrac{e^{\mathrm{i}\omega}}{1 + \mathrm{i}\omega}$

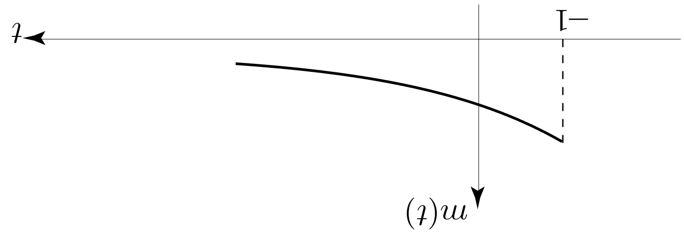
Your solution

(i) Using the frequency shift property with $\omega_0 = 2\pi$

$$l(t) = \mathcal{F}^{-1}\{L(\omega)\} = p_3(t)e^{i2\pi t}$$

(ii) Using the time shift property with $t_0 = -1$

$$m(t) = e^{-(t+1)}u(t+1)$$



4. Further properties of the Fourier Transform

We state these properties without proof. As usual $F(\omega)$ denotes the Fourier Transform of $f(t)$.

(a) Time differentiation property:

$$\mathcal{F}\{f'(t)\} = i\omega F(\omega)$$

(Differentiating a function is said to amplify the higher frequency components because of the additional multiplying factor ω).

(b) Frequency differentiation property:

$$\mathcal{F}\{tf(t)\} = i\frac{dF}{d\omega} \qquad \text{or} \qquad \mathcal{F}\{(-it)f(t)\} = \frac{dF}{d\omega}$$

Note the symmetry between properties (a) and (b).

(c) Duality property:

If

$$\mathcal{F}\{f(t)\} = F(\omega)$$

then

$$\mathcal{F}\{F(t)\} = 2\pi f(-\omega).$$

Informally, the duality property states that we can, apart from the 2π factor, interchange the time and frequency domains provided we put $-\omega$ rather than ω in the second term, this corresponding to a reflection in the vertical axis. If $f(t)$ is even this latter is irrelevant.

Example We know that if

$$f(t) = p_1(t) \quad = \quad \begin{cases} 1 & -1 < t < 1 \\ 0 & \text{otherwise} \end{cases} ,$$

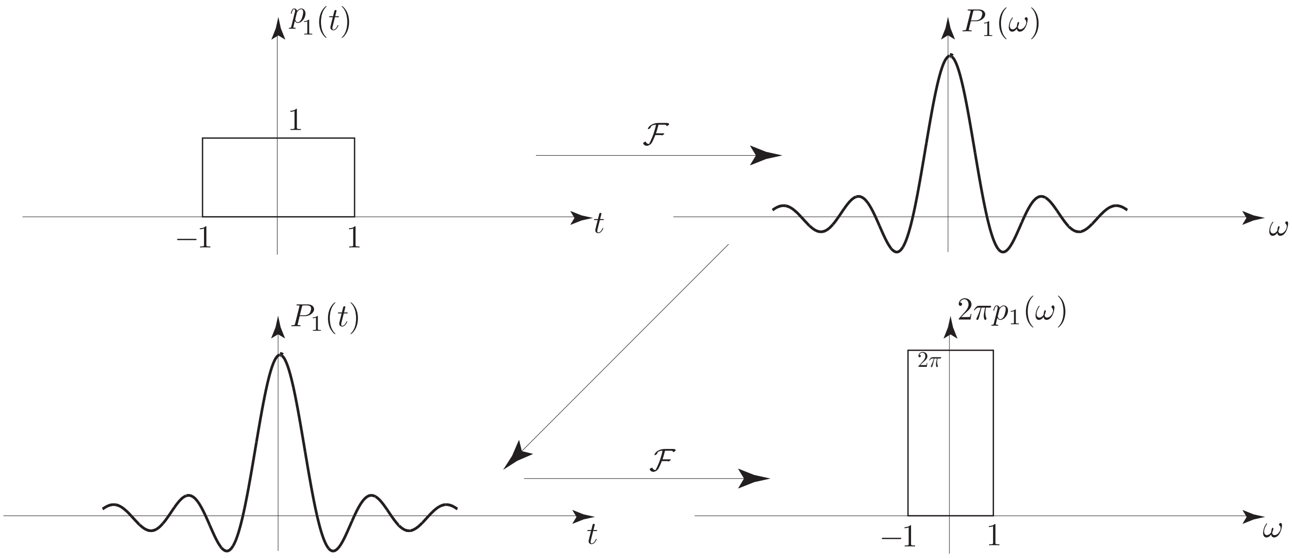
then $$F(\omega) \quad = \quad 2\frac{\sin\omega}{\omega}.$$

Then, by the duality property,

$$\mathcal{F}\{2\frac{\sin t}{t}\} = 2\pi p_1(-\omega) = 2\pi p_1(\omega)$$

(since $p_1(\omega)$ is even).

Graphically



Recalling the Fourier Transform pair

$$f(t) = \begin{cases} e^{-2t} & t > 0 \\ e^{2t} & t < 0 \end{cases} \qquad \text{(or} \quad f(t) = e^{-2|t|} \quad \text{for short)}$$

$$F(\omega) = \frac{4}{4+\omega^2},$$

obtain the Fourier Transforms of

(i) $g(t) = \dfrac{1}{4+t^2}$ (ii) $h(t) = \dfrac{1}{4+t^2}\cos 2t$.

Your solution for (i). Use the linearity and duality properties.

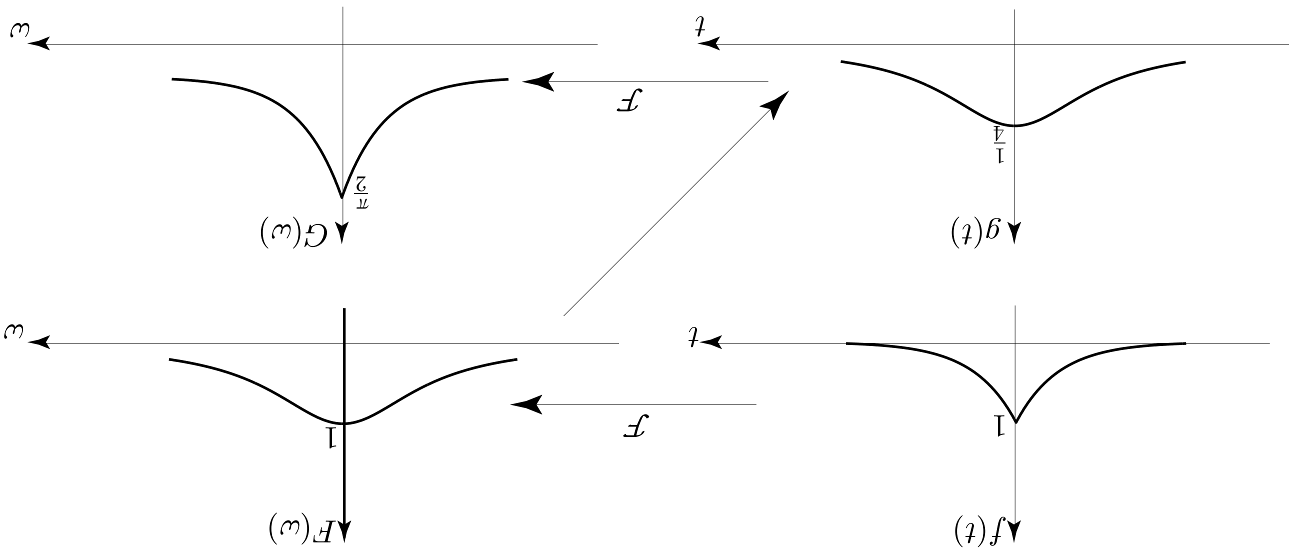
Your solution

We have

$$\mathcal{F}\{f(t)\} \equiv \mathcal{F}\{e^{-2|t|}\} = \frac{4}{4+\omega^2}.$$

$$\therefore \quad \mathcal{F}\{\frac{1}{4}e^{-2|t|}\} = \frac{1}{4+\omega^2} \qquad \text{(by linearity)}$$

$$\therefore \quad \mathcal{F}\{\frac{1}{4+t^2}\} = 2\pi\frac{1}{4}e^{-2|-\omega|} = \frac{\pi}{2}e^{-2|\omega|} = G(\omega) \qquad \text{(by duality).}$$



Your solution for (ii) using the modulation property based on the frequency shift property.

Your solution

We have $h(t) = g(t)\cos 2t$.

$$\therefore \quad \mathcal{F}\{g(t)\cos\omega_0 t\} = \frac{1}{2}(G(\omega-\omega_0) + G(\omega+\omega_0)),$$

so with $\omega_0 = 2$

$$\mathcal{F}\{h(t)\} = \frac{\pi}{4}\left\{e^{-2|\omega-2|} + e^{-2|\omega+2|}\right\} = H(\omega)$$